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Multinationals, R&D and productivity: evidence for UK manufacturing firms

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Idiosyncratic preferences on weighted networks*

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Abstract

We study a static binary-action game on a network in which each agent has an idiosyncratic preference over the two actions and weighs each neighbour in proportion to that neighbour's own connectivity. An agent's best response depends on her environment through a single statistic, the share of her attention directed at neighbours who share her preference, and the threshold it must clear is common to all agents regardless of degree. We give necessary and sufficient conditions for the existence of homogeneous and fully satisfactory equilibria in each of the three regions of the parameter space. Both may fail at once, and in anticoordination games the usual monotonicity arguments do not apply. We show that connectivity weighting endows the game with a weighted potential, so that a pure-strategy equilibrium exists on every network and in every region, and give a three-agent example showing that normalization of the weights alone does not suffice for this conclusion.

Keywords: Network games; weighted networks; idiosyncratic preferences; threshold models; potential games; Nash equilibrium.

JEL classification: C62; C72; D85.

1 Introduction

The payoff parameters divide into three regimes. Where the coordination motive is strong the agent follows her own preference provided enough of her attention is directed at like-minded neighbours; where the differentiation motive is strong she follows it provided not too much is; and between the two lies a band in which acting on one's preference is dominant. Within this common structure we study the two configurations on which the literature has concentrated: the uniform conventions, in which every agent takes the same action irrespective of her preference, and the fully satisfactory profile, in which every agent acts on her own. For each we obtain necessary and sufficient conditions, and in both cases the condition is a requirement

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on the affinity weight of every agent, bounded below where actions are complements and above where they are substitutes. The requirement is uniform across agents, so that what distinguishes one agent's position from another's is the composition of her neighbourhood and not its size.

These conditions describe named profiles but say nothing about existence when those profiles fail, and the gap is not innocuous. In the differentiation regime no uniform convention is ever an equilibrium, whatever the network, and the fully satisfactory profile is ruled out as soon as a single agent is surrounded entirely by like-minded neighbours. Both focal configurations may therefore disappear at once, and the standard route to existence is unavailable: where actions are complements the best-response correspondence is monotone and a fixed point follows from lattice arguments, but where they are substitutes it is antitone and the same arguments yield a cycle rather than an equilibrium. We resolve the question by showing that the connectivity weighting endows the game with a weighted potential, so that the agents' heterogeneous incentives are all generated by a single function of the action profile, each agent reading it on her own scale. A pure-strategy equilibrium then exists on every network, for every distribution of preferences and in every regime. The two halves of the analysis are complementary rather than redundant: the equilibrium conditions identify particular configurations and require only that attention be normalised, while the potential guarantees existence in general but requires the connectivity structure of the weights and identifies no equilibrium in particular. A three-agent example shows that the second hypothesis cannot be weakened to the first, exhibiting a normalised system of weights under which an anticoordination game has no pure-strategy equilibrium at all.

The model is closest to Hernández et al. (2013) and Hernández et al. (2017), who study binary network games with conflicting intrinsic preferences and obtain conditions for the existence of uniform and fully satisfactory equilibria, and to Orlova (2022), who extends the analysis to a broad class of games with idiosyncratic preferences. What separates the present framework from all three is that payoffs here are not semi-anonymous: an agent responds not to how many of her neighbours choose each action but to which of them do, weighted by their own connectivity. This is what makes the affinity weight, rather than a count of neighbours, the relevant statistic, and it is also what makes the existence question non-trivial, since weighting an agent's neighbours by their own position renders the matrix of influence asymmetric.

The paper is organized as follows. Section 2 presents the model: the weighted network, the multiplicative utility function, and the conditions on the weights ω_j^i. Section 3 develops the equilibrium analysis. Lemmas 1, 2 and 3 characterize the best-response function in the three regions of the parameter space; Proposition 4 establishes the conditions under which homogeneous equilibria exist; Propositions 5 and 6 give necessary and sufficient conditions for the fully satisfactory equilibrium in each region; and Theorem 1 and Corollary 7 establish the existence of a pure-strategy equilibrium on every network, together with an example showing that the normalization of the weights alone does not suffice for this conclusion. Section 4 concludes and sets out directions for further work.

2 Model

We consider a static game among n players, each represented by a node in a network. The network is an undirected graph in which each player i has a set of neighbours N_i, with $|N_i| \geq 2$ assumed[1] for all i. Each player has an action set $x_i \in \{0, 1\}$, together with a preference for one of the two actions. We call this preference agent i's *type* $\theta_i \in \{0, 1\}$. Player i's utility is given by:

$$u_i(x_1, \ldots, x_n) = \left(\delta \sum_{j \in N_i} \omega_j^i \mathbf{1}_{\{x_i = x_j\}} + (1 - \delta) \sum_{j \in N_i} \omega_j^i \mathbf{1}_{\{x_i \neq x_j\}} \right) \left(1 + \lambda \mathbf{1}_{\{x_i = \theta_i\}}\right) \tag{1}$$

where $\delta \in [0, 1]$ represents the relative preference for coordination; $\lambda > 0$ measures the intensity with which the player values acting in accordance with her type; and ω_j^i represents the weight that player i assigns to neighbour j. The weights ω_j^i reflect the importance that player i assigns to each neighbour j, considering that we remove the reciprocal link. This adjustment ensures that the weights reflect the net connectivity of j from i's perspective, and are defined as follows:

$$\omega_j^i = \begin{cases} \dfrac{|N_j| - 1}{\sum_{k \in N_i} (|N_k| - 1)} & \text{if } j \in N_i \\ 0 & \text{if } j \notin N_i \text{ or } j = i \end{cases} \tag{2}$$

The first property we obtain regarding the weights is that they are normalized, that is:

$$\sum_{j \in N_i} \omega_j^i = 1. \tag{3}$$

It will be convenient to write $f_j = |N_j| - 1$ for the net connectivity of agent j and $F_i = \sum_{k \in N_i} f_k$ for its total over i's neighbourhood, so that

$$\omega_j^i = \frac{f_j}{F_i} \qquad \text{for } j \in N_i. \tag{4}$$

The assumption $|N_i| \geq 2$ for all i guarantees $f_i \geq 1$ and hence $F_i > 0$. Expression (4) makes visible a feature of the weights that the normalization alone does not convey, *i.e.*, the influence of j on i depends on j only, and i enters through the constant that scales her own attention to sum to one.

That is, a neighbour carries greater weight the more connected she is herself. The parameter δ governs this preference: when δ is high, the player prefers to coordinate; when it is low, she prefers to differentiate. This term captures strategic interaction with neighbours. The second factor, $(1 + \lambda \mathbf{1}_{\{x_i = \theta_i\}})$, amplifies utility if the player acts in accordance with her type.

[1] The assumption $|N_i| \geq 2$ ensures that every agent's weight vector is well defined and non-degenerate.

3 Nash Equilibrium

In this section we proceed to characterize the Nash equilibria of the n-agent game with complete information. A pure-strategy profile $x^* = (x_1^*, \ldots, x_n^*)$ is a Nash equilibrium if $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*)$ for every player i and every $x_i \in \{0,1\}$, where x_{-i} denotes the action profile of all agents except i. The analysis proceeds in three steps. We first partition the parameter space (δ, λ) into three regions, coordination, anticoordination, and dominant action, and derive the best-response function in each of them (Lemmas 1–3). The best response is governed in every region by a single threshold $\tau_{\delta,\lambda}$, common to all agents regardless of their degree or the distribution of weights within their neighbourhood. We then study two polar types of equilibrium: *homogeneous* equilibria, in which all agents choose the same action irrespective of their type, and *fully satisfactory* equilibria, in which every agent chooses the action dictated by her own type. Proposition 4 establishes existence conditions for homogeneous equilibria in each region, while Propositions 5 and 6 provide necessary and sufficient conditions for the fully satisfactory equilibrium in coordination and anticoordination games, respectively.

We define the following regions of the parameter space (δ, λ):

Coordination region R_C:

$$\mathcal{R}_C \;=\; \left\{(\delta,\lambda):\; \delta \;>\; \frac{1+\lambda}{2+\lambda}\right\},$$

Anticoordination region R_A:

$$\mathcal{R}_A \;=\; \left\{(\delta,\lambda):\; \delta \;<\; \frac{1}{2+\lambda}\right\},$$

Dominant-action region R_D:

$$\mathcal{R}_D \;=\; \left\{(\delta,\lambda):\; \frac{1}{2+\lambda} \;<\; \delta \;<\; \frac{1+\lambda}{2+\lambda}\right\}.$$

The three regions are mutually exclusive and exhaustive up to the measure-zero sets $\delta = \frac{1}{2+\lambda}$ and $\delta = \frac{1+\lambda}{2+\lambda}$. The symmetry of the partition is notable: the two thresholds delimiting $\mathcal{R}_D$ satisfy $\frac{1}{2+\lambda} + \frac{1+\lambda}{2+\lambda} = 1$, so that the intervals $\mathcal{R}_A$ and $\mathcal{R}_C$ are symmetric about $\delta = \frac{1}{2}$, and $\mathcal{R}_D$ expands with λ: as $\lambda \to \infty$ it covers all of $(0,1)$, and as $\lambda \to 0$ it collapses to the point $\{\frac{1}{2}\}$. Figure 1 shows the three regions.

In what follows, we study the conditions for the best response of a player i as a function of her type θ_i.

3.1 Best-response functions

For every action profile $\mathbf{x} \in \{0,1\}^N$, we define

$$S^i_{\theta_i} \;=\; \sum_{j \in N_i} \omega^i_j \, \mathbf{1}_{\{x_j = \theta_i\}} \qquad \text{and} \qquad S^i_{1-\theta_i} \;=\; \sum_{j \in N_i} \omega^i_j \, \mathbf{1}_{\{x_j \neq \theta_i\}},$$

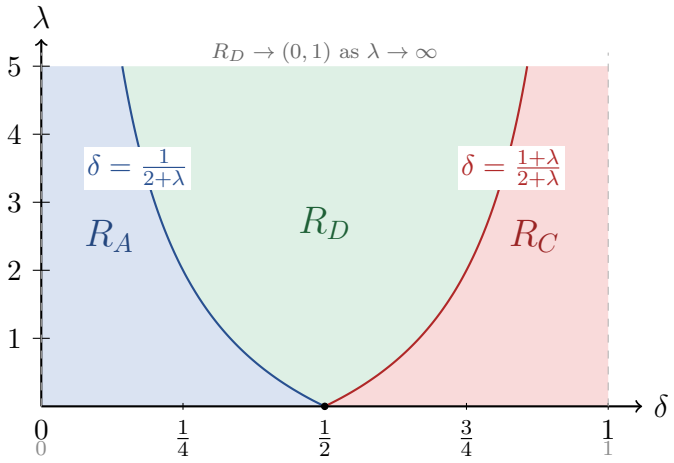


Figure 1: Partition of the parameter space $(\delta, \lambda) \in (0,1) \times (0,\infty)$ into the three regions of the game. The blue region R_A corresponds to anticoordination games, the green region R_D to dominant-action games, and the red region R_C to coordination games. The two boundaries meet at the point $(\frac{1}{2}, 0)$ and separate symmetrically about $\delta = \frac{1}{2}$ as λ increases.

that is, $S^i_{\theta_i}$ is the accumulated weight of the neighbours whose action coincides with i's preference, while $S^i_{1-\theta_i}$ is the accumulated weight of those whose action differs from it. By the normalization condition (3), $S^i_{\theta_i} + S^i_{1-\theta_i} = 1$ whenever $|N_i| \geq 2$.

We define the conformity-adjusted coordination threshold $\tau_{\delta,\lambda}$:

$$\tau_{\delta,\lambda} \;=\; \frac{\delta(2+\lambda) - (1+\lambda)}{(2\delta - 1)(2+\lambda)}. \tag{5}$$

Lemma 1 (Best response. Coordination games)**.** *In a game $(\delta, \lambda) \in R_C$, the best-response function of agent $i \in N$ with preference θ_i is:*

$$BR_i\Big(\theta_i,\, S^i_{\theta_i}\Big) \;=\; \begin{cases} \theta_i, & \text{if } S^i_{\theta_i} \geq \tau_{\delta,\lambda}, \\ 1-\theta_i, & \text{if } S^i_{\theta_i} < \tau_{\delta,\lambda}, \end{cases}$$

Proof. Fix an agent $i \in N$ with preference $\theta_i \in \{0,1\}$, and consider a unilateral deviation from $x_i = \theta_i$ to $x_i = 1 - \theta_i$, holding fixed the action profile of the rest of the agents. Since $S^i_{\theta_i} + S^i_{1-\theta_i} = 1$, agent i's utility under each action is:

$$u_i(\theta_i,\, \mathbf{x}_{-i}) = \Big(\delta\, S^i_{\theta_i} + (1-\delta)\, S^i_{1-\theta_i}\Big)(1+\lambda),$$
$$u_i(1-\theta_i,\, \mathbf{x}_{-i}) = \delta\, S^i_{1-\theta_i} + (1-\delta)\, S^i_{\theta_i},$$

where the factor $(1+\lambda)$ in the first expression reflects the conformity bonus that the agent obtains from acting in accordance with her preference θ_i.

Agent i prefers $x_i = \theta_i$ if and only if $u_i(\theta_i, \mathbf{x}_{-i}) \geq u_i(1-\theta_i, \mathbf{x}_{-i})$, that is:

$$\Big(\delta\, S^i_{\theta_i} + (1-\delta)\, S^i_{1-\theta_i}\Big)(1+\lambda) \;\geq\; \delta\, S^i_{1-\theta_i} + (1-\delta)\, S^i_{\theta_i}.$$

Substituting $S^i_{1-\theta_i} = 1 - S^i_{\theta_i}$ and expanding:

$$\begin{aligned}\big[\delta\, S^i_{\theta_i} + (1-\delta)(1 - S^i_{\theta_i})\big](1+\lambda) \;&\geq\; \delta(1 - S^i_{\theta_i}) + (1-\delta)\, S^i_{\theta_i} \\ \big[(2\delta - 1)\, S^i_{\theta_i} + (1-\delta)\big](1+\lambda) \;&\geq\; \delta - (2\delta - 1)\, S^i_{\theta_i}.\end{aligned}$$

Grouping the terms containing $S^i_{\theta_i}$:

$$(2\delta - 1)(1+\lambda)\, S^i_{\theta_i} + (2\delta - 1)\, S^i_{\theta_i} \;\geq\; \delta - (1-\delta)(1+\lambda).$$

Factoring the left-hand side:

$$(2\delta - 1)(2+\lambda)\, S^i_{\theta_i} \;\geq\; \delta - (1-\delta)(1+\lambda) \;=\; \delta(2+\lambda) - (1+\lambda).$$

Since $(\delta, \lambda) \in R_C$ implies $\delta > \frac{1}{2}$, we have $(2\delta - 1)(2 + \lambda) > 0$, so it is valid to divide both sides without reversing the inequality:

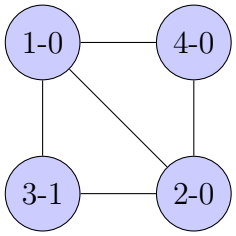
$$S^i_{\theta_i} \;\geq\; \frac{\delta(2+\lambda) - (1+\lambda)}{(2\delta - 1)(2+\lambda)} \;=:\; \tau_{\delta,\lambda}.$$

It is straightforward to verify that in region R_C we have $\tau_{\delta,\lambda} \in (0, \frac{1}{2})$: the numerator is positive since $\delta > \frac{1}{2}$ and $\delta > \frac{1+\lambda}{2+\lambda}$ implies $\delta(2+\lambda) > 1 + \lambda$; the denominator is positive since $\delta > \frac{1}{2}$; and $\tau_{\delta,\lambda} < \frac{1}{2}$ follows from $\lambda > 0$. This confirms that the threshold is non-trivial within the coordination region.

Therefore, $x_i = \theta_i$ is the best response if and only if $S^i_{\theta_i} \geq \tau_{\delta,\lambda}$. Otherwise, $x_i = 1 - \theta_i$ yields strictly higher utility. We adopt the standard tie-breaking rule: in case of exact indifference ($S^i_{\theta_i} = \tau_{\delta,\lambda}$), the agent chooses her preferred action θ_i. □

When δ exceeds the threshold $\frac{1+\lambda}{2+\lambda}$, the marginal benefit of coordinating with an additional neighbour dominates the benefit of acting in accordance with one's own type. In this region, strategic complementarity is intense enough that no agent prefers to deviate from the majority of her neighbourhood, regardless of her type θ_i. The conformity incentive λ raises the threshold on δ required to enter R_C. The stronger the intensity of individual preference, the higher the coordination advantage must be for the agent to place it above her type.

Let us now consider an example illustrating the best-response conditions of Lemma 1. Figure 2 shows a four-node network and its associated weight matrix. We fix $\lambda = 1$ and each node's idiosyncratic preferences θ_i, where $\{\theta_1 = 0,\ \theta_2 = 0,\ \theta_3 = 1,\ \theta_4 = 0\}$. Taking $\delta = 0.7$, which satisfies the conditions $\delta > \frac{1}{2}$ and $\delta > \frac{1+\lambda}{2+\lambda}$, we obtain a coordination threshold equal to $\tau_{\delta,\lambda} = 0.083$. Likewise, assuming each node takes the action equal to its preference, we obtain $\left\{S^1_{\theta_1} = \frac{3}{4},\ S^2_{\theta_2} = \frac{3}{4},\ S^3_{\theta_3} = 0,\ S^4_{\theta_4} = 1\right\}$. Given this threshold, the marginal benefit that player 3 obtains from coordinating her action with her neighbours' dominates the benefit derived from acting in accordance with her type. Consequently, player 3 has an incentive to coordinate with the rest of the network, even though her idiosyncratic preference differs from the action adopted by her neighbours. In contrast, none of the remaining nodes has an incentive to deviate from her strategy, since in all cases $S^i_{\theta_i} > \tau_{\delta,\lambda}$ holds. To facilitate understanding of the examples throughout the chapter, the value of the parameter λ, the network's weight matrix W, the individual preferences θ_i, and the weights of neighbours



(a) Network with idiosyncratic preferences

$$W = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \end{pmatrix}$$

(b) Weight matrix

Figure 2: Coordination games: example of a network with weighted interactions and idiosyncratic preferences. The first value numbers the node and the second shows each node's preference. Colour represents the action taken by each node, where blue nodes take action 0 and red nodes take action 1.

whose action coincides with each node's preference, $S^i_{\theta_i}$, will remain constant across all of the examples analysed throughout this paper .

Following the same method of computation, Lemma 2 gives the best responses in the region corresponding to anticoordination games. To this end, we first define the corresponding anticoordination threshold:

$$\tau^A_{\delta,\lambda} \;:=\; 1 - \tau_{\delta,\lambda} \;=\; \frac{\delta(2+\lambda) - 1}{(2\delta - 1)(2+\lambda)}. \tag{6}$$

Lemma 2 (Best response. Anticoordination games)**.** *In a game $(\delta, \lambda) \in \mathcal{R}_A$, the best-response function of agent $i \in N$ with preference θ_i is:*

$$\mathrm{BR}_i\Big(\theta_i,\, S^i_{1-\theta_i}\Big) \;=\; \begin{cases} \theta_i, & \text{if } S^i_{1-\theta_i} \;\geq\; \tau^A_{\delta,\lambda}, \\ 1-\theta_i, & \text{if } S^i_{1-\theta_i} \;<\; \tau^A_{\delta,\lambda}. \end{cases}$$

Proof. Fix $(\delta, \lambda) \in \mathcal{R}_A$, so that $\delta < \frac{1}{2+\lambda}$, which implies $\delta < \frac{1}{2}$ and $\lambda < \frac{1-2\delta}{\delta}$. The condition $u_i(\theta_i, x_{-i}) \geq u_i(1-\theta_i, x_{-i})$ is algebraically identical to that of Lemma 1, leading to the inequality:

$$(2\delta - 1)(2+\lambda)\, S^i_{\theta_i} \;\geq\; \delta(2+\lambda) - (1+\lambda).$$

In $\mathcal{R}_A$ we have $(2\delta - 1)(2+\lambda) < 0$; dividing by this negative quantity reverses the direction of the inequality:

$$S^i_{\theta_i} \;\leq\; \frac{\delta(2+\lambda) - (1+\lambda)}{(2\delta-1)(2+\lambda)} \;=\; \tau_{\delta,\lambda}.$$

Expressing the condition in terms of $S^i_{1-\theta_i} = 1 - S^i_{\theta_i}$ we obtain:

$$1 - S^i_{1-\theta_i} \;\leq\; \tau_{\delta,\lambda} \;\iff\; S^i_{1-\theta_i} \;\geq\; 1 - \tau_{\delta,\lambda} \;=:\; \tau^A_{\delta,\lambda}.$$

We verify that $\tau^A_{\delta,\lambda} \in (0, \frac{1}{2})$ in $\mathcal{R}_A$. The numerator $\delta(2+\lambda) - 1 < 0$ and the denominator $(2\delta-1)(2+\lambda) < 0$ in $\mathcal{R}_A$, so $\tau^A > 0$. For the upper bound, $\tau^A < \frac{1}{2}$ is equivalent to the negative denominator satisfying $(2\delta-1)(2+\lambda) < 2[\delta(2+\lambda) - 1]$, which reduces to $\lambda > 0$, satisfied by hypothesis. Therefore, agent i prefers $x_i = \theta_i$ if and only if $S^i_{1-\theta_i} \geq \tau^A_{\delta,\lambda}$. The standard tie-breaking rule assigns θ_i under exact indifference. □

When δ falls below $\frac{1}{2+\lambda}$, the agent obtains greater utility from differentiating herself from her neighbours than from coordinating with them. Strategic substitutability is now the dominant force: the incentive for differentiation exceeds both the incentive for coordination and the incentive for acting in accordance with type, so the agent prefers the opposite action to that of the majority of her neighbourhood. Region R_A shrinks as λ grows, since stronger individual preferences make departing from one's own type more costly.

Following the logic of Figure 2, Figure 3 shows a four-node network in which, based on Lemma 2, we obtain the best-response actions. Taking a parameter value $\delta = 0.2$, which satisfies the conditions $\delta < \frac{1}{2}$ and $\delta < \frac{1}{2+\lambda}$, we obtain a coordination threshold given by $1 - \tau_{\delta,\lambda} = 0.22$. Given this threshold, the marginal benefit that player 4 obtains from differentiating her action from her neighbours' dominates the benefit associated with acting in accordance with her type. Consequently, player 4 has an incentive to deviate and choose an action different from that of her network neighbours. In contrast, the remaining nodes have no incentive to deviate from their strategy, since in all cases $S^i_{\theta_i} > \tau^A_{\delta,\lambda}$ holds.

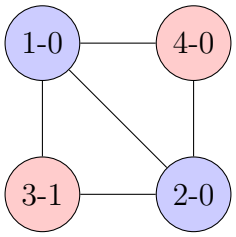


Figure 3: Anticoordination games: example of a network with weighted interactions and idiosyncratic preferences. The first value numbers the node and the second shows each node's preference. Colour represents the action taken by each node, where blue nodes take action 0 and red nodes take action 1.

Lemma 3 (Best response. Dominant-action games)**.** *In a game $(\delta, \lambda) \in R_D$, the best-response function of agent $i \in N$ with preference θ_i is:*

$$BR_i\big(\theta_i,\, S^i_{\theta_i}\big) \;=\; \theta_i \qquad \textit{for all } S^i_{\theta_i} \in [0,1].$$

Proof. Fix $i \in N$ and let $\theta_i \in \{0,1\}$ denote her preference. Starting from the utility expressions obtained in the proof of Lemma 1, the difference $u_i(\theta_i, x_{-i}) - u_i(1-\theta_i, x_{-i})$ reduces to:

$$\Delta_i(S^i_{\theta_i}) \;:=\; (2\delta - 1)(2+\lambda)\, S^i_{\theta_i} \;-\; \big[\delta(2+\lambda) - (1+\lambda)\big].$$

We will show that $\Delta_i(S^i_{\theta_i}) > 0$ for all $S^i_{\theta_i} \in [0,1]$, which establishes that θ_i is a strictly dominant strategy. Region R_D contains two subcases depending on the sign of $2\delta - 1$:

Subcase (a): $\delta > \frac{1}{2}$ (and $\delta < \frac{1+\lambda}{2+\lambda}$). The coefficient $(2\delta-1)(2+\lambda) > 0$ makes Δ_i minimal at $S^i_{\theta_i} = 0$:

$$\min_{S^i_{\theta_i} \in [0,1]} \Delta_i \;=\; -\big[\delta(2+\lambda) - (1+\lambda)\big] \;=\; (1+\lambda) - \delta(2+\lambda) \;>\; 0,$$

where the inequality is equivalent to $\delta < \frac{1+\lambda}{2+\lambda}$, which is satisfied in this subcase.

Subcase (b): $\delta < \frac{1}{2}$ (and $\delta > \frac{1}{2+\lambda}$). The coefficient $(2\delta - 1)(2 + \lambda) < 0$ makes Δ_i minimal at $S^i_{\theta_i} = 1$:

$$\min_{S^i_{\theta_i} \in [0,1]} \Delta_i \;=\; (2\delta - 1)(2 + \lambda) - \big[\delta(2 + \lambda) - (1 + \lambda)\big] \;=\; \delta(2 + \lambda) - 1 \;>\; 0,$$

where the inequality is equivalent to $\delta > \frac{1}{2+\lambda}$, which is satisfied in this subcase.

In both subcases, $\Delta_i(S^i_{\theta_i}) > 0$ for all $S^i_{\theta_i} \in [0, 1]$, regardless of the action profile x_{-i} and of the network's weighted topology. Therefore, θ_i is i's unique best response. □

Figure 4 illustrates the dominant-action region using the same four-node network and preference profile as in the preceding examples. Fixing $\delta = 0.6$, which satisfies $\frac{1}{2+\lambda} < \delta < \frac{1+\lambda}{2+\lambda}$, the utility difference $\Delta_i(S^i_{\theta_i}) = 0.6\, S^i_{\theta_i} + 0.2$ is strictly positive for every node in the network, confirming that each agent's type is a strictly dominant strategy and no node has an incentive to deviate.

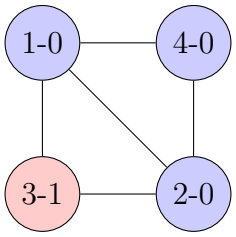


Figure 4: Dominant-action games: example of a network with weighted interactions and idiosyncratic preferences. The first value numbers the node and the second shows each node's preference. Colour represents the action taken by each node, where blue nodes take action 0 and red nodes take action 1.

In the intermediate band, the conformity incentive λ is sufficiently high relative to the net advantage of coordination or anticoordination that no collective action by one's neighbours can overturn one's choice of type: the agent chooses θ_i regardless of what her neighbourhood does. This region grows monotonically with λ and collapses to a measure-zero set as $\lambda \to 0$, reflecting that, absent individual preferences, the choice of action depends entirely on the coordination or anticoordination structure. The label "dominant" is precise in the strict sense: θ_i is a strictly dominant strategy for every agent i on any network when $(\delta, \lambda) \in R_D$.

Unlike Lemmas 1 and 2, Lemma 3 defines no non-trivial threshold. This reflects the fact that in R_D, the individual preference θ_i dominates any neighbourhood configuration, including the extreme case in which every neighbour chooses the opposite action. In particular, the threshold $\tau_{\delta,\lambda}$ satisfies $\tau_{\delta,\lambda} \leq 0$ in subcase (a) and $\tau_{\delta,\lambda} \geq 1$ in subcase (b), which makes the condition of Lemma 1 trivially satisfied throughout region R_D.

3.2 Nash equilibria: typology

Following the terminology of (Hernández et al., 2013), we can distinguish between different types of equilibria. We next present the definitions of *homogeneous* equilibria and *fully satisfactory* equilibria.

Definition 1. *A Nash equilibrium x^* is* homogeneous *if $x_i^* = x_j^*$ for all $i, j \in N$.*

Action x_i is *satisfactory* for agent i if $x_i = \theta_i$.

Definition 2. *A Nash equilibrium x^* is* fully satisfactory *if $x_i^* = \theta_i$ for all $i \in N$, that is, $x^* = \bar{\theta}$.*

3.2.1 Homogeneous equilibria

The following proposition establishes sufficient conditions for the existence of homogeneous equilibria in coordination, anticoordination, and dominant-action games.

Proposition 4. *Let $(G, \bar{\theta})$ be an arbitrary network where $|N_i| \geq 2$ for all i.*

1. ***Coordination games, $\mathcal{R}_C$.*** *The profiles $x^* = (0, \ldots, 0)$ and $x^* = (1, \ldots, 1)$ are Nash equilibria for every network $(G, \bar{\theta})$.*

2. ***Anticoordination games, $\mathcal{R}_A$.*** *No homogeneous profile is a Nash equilibrium for any network $(G, \bar{\theta})$.*

3. ***Dominant-action games, $\mathcal{R}_D$.*** *A homogeneous profile $x^* \in \{(0, \ldots, 0), (1, \ldots, 1)\}$ is a Nash equilibrium if and only if the preference profile is itself homogeneous with the same value. In particular, in this case $x^* = \bar{\theta}$ is also the fully satisfactory profile and constitutes the unique Nash equilibrium of the network.*

Proof. Part (1). Consider the profile $x^* = (1, \ldots, 1)$; the argument for $x^* = (0, \ldots, 0)$ is analogous. Under this profile, every neighbour of i chooses action 1, so $S^i_{\theta_i} = \mathbf{1}_{\{\theta_i = 1\}}$. We distinguish two cases according to the agent's type.

1. Suppose $\theta_i = 1$. Then $S^i_{\theta_i} = 1 \geq \tau_{\delta,\lambda}$, since $\tau_{\delta,\lambda} < \frac{1}{2} < 1$ in $\mathcal{R}_C$. By Lemma 1, $\mathrm{BR}_i = \theta_i = 1 = x_i^*$. The agent has no incentive to deviate.

2. Suppose $\theta_i = 0$. Then $S^i_{\theta_i} = 0 < \tau_{\delta,\lambda}$, since $\tau_{\delta,\lambda} > 0$ in $\mathcal{R}_C$. By Lemma 1, $\mathrm{BR}_i = 1 - \theta_i = 1 = x_i^*$. The agent prefers action 1 precisely because all of her neighbours choose it, and the coordination incentive dominates the individual preference incentive. This case is illustrated in the example of Figure 2.

Part (2). Consider $x^* = (1, \ldots, 1)$; the argument for $x^* = (0, \ldots, 0)$ is symmetric. Under this profile, every neighbour of i chooses 1, so $S^i_{1-\theta_i} = \mathbf{1}_{\{1-\theta_i = 1\}} = \mathbf{1}_{\{\theta_i = 0\}}$.

1. Suppose $\theta_i = 0$. Then $1 - \theta_i = 1$ and $S^i_{1-\theta_i} = 1 \geq \tau^A_{\delta,\lambda}$, since $\tau^A < \frac{1}{2} < 1$. By Lemma 2, $\mathrm{BR}_i = \theta_i = 0 \neq 1 = x_i^*$, that is, the agent deviates to action 0.

2. Suppose $\theta_i = 1$. Then $1 - \theta_i = 0$ and $S^i_{1-\theta_i} = 0 < \tau^A_{\delta,\lambda}$. By Lemma 2, $\mathrm{BR}_i = 1 - \theta_i = 0 \neq 1 = x_i^*$; the agent likewise deviates to 0 because the total absence of neighbours anticoordinating with her makes her anticoordination incentive dominate her preference incentive. This case is illustrated in the example of Figure 3.

In both cases, some agent has an incentive to deviate from $x^* = (1, \ldots, 1)$, regardless of the profile $\bar{\theta}$. The argument for $x^* = (0, \ldots, 0)$ is identical by symmetry.

Part (3). In $\mathcal{R}_D$, action θ_i is a strictly dominant strategy for every $i \in N$: for any actions of her neighbours, $u_i(\theta_i, x_{-i}) > u_i(1-\theta_i, x_{-i})$. This follows from the fact that the minimum of $(2\delta-1)(2+\lambda)S^i_{\theta_i} - [\delta(2+\lambda) - (1+\lambda)]$ over $S^i_{\theta_i} \in [0,1]$ is strictly positive in $\mathcal{R}_D$:

- For $\delta > \frac{1}{2}$ (and $\delta < \frac{1+\lambda}{2+\lambda}$), the coefficient $(2\delta-1)(2+\lambda) > 0$ makes the minimum attained at $S^i_{\theta_i} = 0$, giving $-[\delta(2+\lambda) - (1+\lambda)] > 0$, equivalent to $\delta < \frac{1+\lambda}{2+\lambda}$.
- For $\delta < \frac{1}{2}$ (and $\delta > \frac{1}{2+\lambda}$), the coefficient $(2\delta-1)(2+\lambda) < 0$ makes the minimum attained at $S^i_{\theta_i} = 1$, giving $\delta(2+\lambda) - 1 > 0$, equivalent to $\delta > \frac{1}{2+\lambda}$.

Since θ_i is dominant, the unique Nash equilibrium is $x^* = \bar{\theta}$ (Proposition 4, region $\mathcal{R}_D$). A homogeneous profile $x^* = (c, \ldots, c)$, $c \in \{0,1\}$, is an equilibrium if and only if $\mathrm{BR}_i = c = \theta_i$ for all i, which requires that every type satisfy $\theta_i = c$, that is, $x^* = \bar{\theta}$. Both the general case and the two subcases are illustrated in Figure 4. □

Proposition 4 reveals a structural asymmetry between the three regions. In R_C, the coordination incentive is intense enough that any agent who deviates from a unanimous profile loses complementarity with the entirety of her neighbourhood, a cost that strictly exceeds the conformity bonus λ. Homogeneous profiles are therefore Nash equilibria regardless of the preference profile and of the network's weighted topology. In R_A, the logic is reversed: anti-coordination causes any agent who shares the action of the homogeneous profile to want to differentiate herself, so homogeneous equilibria are universally destroyed. In R_D, the dominant strategy of Lemma 3 ensures that each agent chooses θ_i regardless of her neighbourhood, so a homogeneous profile can be an equilibrium only if all preferences coincide. A notable feature of our weighted framework, absent from the unweighted models of Hernández et al. (2013) and Hernández et al. (2017), is that the homogeneous-equilibrium condition in R_C is entirely independent of degree and of the distribution of weights. The same conclusion holds for every network topology.

3.2.2 Fully satisfactory equilibria

To analyse the fully satisfactory equilibrium, we introduce the following parameter, which will serve as a statistic for verifying these conditions.

For each agent $i \in N$, we define the *affinity weight* as:

$$\Omega_i^+ \;=\; \sum_{j \in N_i} \omega_j^i \, \mathbf{1}_{\{\theta_j = \theta_i\}}, \tag{7}$$

that is, Ω_i^+ is the accumulated weight that agent i assigns to the set of neighbours who share her preference. Note that, under the fully satisfactory profile $x^* = \bar{\theta}$, $S^i_{\theta_i} = \Omega_i^+$ for all $i \in N$, since every neighbour chooses precisely her preferred action. Symmetrically, $S^i_{1-\theta_i} = 1 - \Omega_i^+$ measures the accumulated weight of neighbours whose preference differs from i's. The quantity Ω_i^+ depends exclusively on the network's weighted topology and on the preference profile $\bar{\theta}$, and constitutes the network's sufficient statistic for determining whether agent i has an incentive to deviate from the profile $x^* = \bar{\theta}$.

Before turning to the equilibrium conditions, it is worth making precise how the affinity weight differs from the simplest alternative measure of an agent's position, namely the proportion of her neighbours who share her preference. Writing $N_i^+ = \{j \in N_i : \theta_j = \theta_i\}$ for that set of neighbours and $h_i = |N_i^+|/|N_i|$ for its share, both magnitudes count the same neighbours and differ only in the unit of account: Ω_i^+ assigns each of them a share proportional to her net connectivity f_j, whereas h_i assigns each an equal share. Using (4) and writing $\bar{f}(A) = |A|^{-1} \sum_{j \in A} f_j$ for the mean net connectivity on a non-empty set A, the numerator of Ω_i^+ is $\sum_{j \in N_i^+} f_j = |N_i^+|\, \bar{f}(N_i^+)$ and its denominator is $F_i = |N_i|\, \bar{f}(N_i)$, so that for any agent with $N_i^+ \neq \emptyset$,

$$\Omega_i^+ \;=\; h_i \cdot \frac{\bar{f}(N_i^+)}{\bar{f}(N_i)}. \tag{8}$$

The affinity weight thus exceeds the proportion of like-minded neighbours when those neighbours are, on average, better connected than the neighbourhood taken as a whole, and falls short of it when they are worse connected; the two coincide when connectivity carries no information about preference within the neighbourhood, and in particular whenever all of i's neighbours have the same degree. The discrepancy is therefore bounded by the dispersion of degrees within the neighbourhood, and the weighted and unweighted readings of an agent's position separate only to the extent that this dispersion is correlated with the distribution of preferences.

Figure 5 illustrates both directions of the comparison on the network used throughout, holding the network fixed and varying only the assignment of preferences. Here $f_1 = f_2 = 2$, $f_3 = f_4 = 1$ and $F_i = 4$ for every agent. Under the profile $\bar{\theta} = (0, 0, 1, 0)$ of panel (a) we obtain $\Omega^+ = \left(\frac{3}{4}, \frac{3}{4}, 0, 1\right)$ against $h = \left(\frac{2}{3}, \frac{2}{3}, 0, 1\right)$, whereas under the profile $\bar{\theta}' = (0, 1, 0, 0)$ of panel (b), obtained by exchanging the preferences of agents 2 and 3, we obtain $\Omega^+ = \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}\right)$ against $h = \left(\frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2}\right)$. The comparison is most informative for agent 1, whose position is identical across the two panels as far as a count of neighbours is concerned: she has three neighbours in both, of whom two share her preference, so $h_1 = \frac{2}{3}$ in each case. Her affinity weight, however, falls from $\frac{3}{4}$ to $\frac{1}{2}$. In panel (a) her like-minded neighbours are agents 2 and 4, with $\bar{f}(N_1^+) = \frac{3}{2}$ above the neighbourhood mean $\bar{f}(N_1) = \frac{4}{3}$, so that (8) gives $\Omega_1^+ = \frac{2}{3} \cdot \frac{3/2}{4/3} = \frac{3}{4}$; in panel (b) they are the two peripheral agents 3 and 4, with $\bar{f}(N_1^+) = 1$ below $\bar{f}(N_1) = \frac{4}{3}$, and $\Omega_1^+ = \frac{2}{3} \cdot \frac{1}{4/3} = \frac{1}{2}$. Agents 3 and 4 in panel (b) illustrate the degenerate case: the neighbour sharing their preference has exactly the neighbourhood's mean connectivity, so weighting is immaterial and $\Omega_i^+ = h_i = \frac{1}{2}$.

Proposition 5 below gives necessary and sufficient conditions for fully satisfactory equilibria in coordination games.

Proposition 5 (Fully satisfactory equilibrium in R_C)**.** *Let $(G, \bar{\theta})$ be an arbitrary network and $(\delta, \lambda) \in R_C$. The fully satisfactory profile $x^* = \bar{\theta}$ is a Nash equilibrium* if and only if

$$\Omega_i^+ \;\geq\; \tau_{\delta,\lambda} \qquad \forall\, i \in N, \tag{9}$$

where $\tau_{\delta,\lambda}$ is as in (5)*.*

Proof. Under the profile $x^* = \bar{\theta}$, every agent $j \in N_i$ chooses $x_j = \theta_j$, so $S^i_{\theta_i} = \Omega_i^+$ for all $i \in N$.



(a) $\bar{\theta} = (0, 0, 1, 0)$ (b) $\bar{\theta}' = (0, 1, 0, 0)$

Figure 5: The network of Figure 2 under two preference profiles. The first value numbers the node and the second shows its preference. Colour here indicates the agent's preference rather than her action, since the comparison between Ω_i^+ and h_i involves only the pair $(G, \bar{\theta})$ and no action profile. Agent 1 has two like-minded neighbours out of three in both panels, but they are the well-connected agents 2 and 4 in panel (a) and the peripheral agents 3 and 4 in panel (b).

We begin with sufficiency. If $\Omega_i^+ \geq \tau_{\delta,\lambda}$ for all $i \in N$, then $S_{\theta_i}^i = \Omega_i^+ \geq \tau_{\delta,\lambda}$ and, by Lemma 1, $BR_i = \theta_i = x_i^*$ for all $i \in N$. Therefore, no agent has a profitable unilateral deviation from $x^* = \bar{\theta}$.

For necessity, suppose there exists $i_0 \in N$ such that $\Omega_{i_0}^+ < \tau_{\delta,\lambda}$. Then $S_{\theta_{i_0}}^{i_0} = \Omega_{i_0}^+ < \tau_{\delta,\lambda}$ and, by Lemma 1, $BR_{i_0} = 1 - \theta_{i_0} \neq x_{i_0}^*$. Agent i_0 therefore has a strict incentive to deviate from the profile $x^* = \bar{\theta}$, which implies that this profile is not a Nash equilibrium.

Consequently, $x^* = \bar{\theta}$ is a Nash equilibrium if and only if condition (9) is satisfied for every $i \in N$. □

Condition (9) admits a direct reading in terms of network structure. The fully satisfactory profile $x^* = \bar{\theta}$ is sustainable if and only if the network is sufficiently homophilous in weighted terms, that is, if every agent's neighbourhood contains enough well-connected neighbours of her own type to exceed the universal threshold $\tau_{\delta,\lambda}$. By (8), what matters is not the raw count of like-minded neighbours but their aggregate connectivity, and the two need not point the same way. A small number of well-connected allies can satisfy the condition where a larger number of peripheral ones cannot, but the reverse is equally possible, and an agent whose like-minded neighbours are the peripheral members of her neighbourhood is in a weaker position than a count of allies would suggest. Since the threshold is the same for every agent regardless of her degree, the equilibrium condition takes the form of a uniform requirement on the affinity weight Ω_i^+. This contrasts with the unweighted models of Hernández et al. (2017) and Orlova (2022), where analogous conditions depend on each agent's degree and therefore vary across agents even for identical action profiles.

For the network developed in Figure 2, the value of Ω_i^+ for each node is $\{\Omega_1^+ = \frac{3}{4}, \Omega_2^+ = \frac{3}{4}, \Omega_3^+ = 0, \Omega_4^+ = 1\}$. Node 3 has an incentive to deviate from her preference to coordinate with the rest of the group. Condition (9) fails to hold since $\Omega_3^+ < \tau_{\delta\lambda}$.

Proposition 6 below gives necessary and sufficient conditions for fully satisfactory equilibria in anticoordination games.

Proposition 6 (Fully satisfactory equilibrium in R_A)**.** *Let $(G, \bar{\theta})$ be an arbitrary network*

and $(\delta, \lambda) \in R_A$. *The fully satisfactory profile* $x^* = \bar{\theta}$ *is a Nash equilibrium* if and only if

$$\Omega_i^+ \;\leq\; 1 - \tau_{\delta,\lambda}^A \qquad \forall\, i \in N, \tag{10}$$

where $\tau_{\delta,\lambda}^A = 1 - \tau_{\delta,\lambda}$ *is the anticoordination threshold.*

Proof. Under the profile $x^* = \bar{\theta}$, $S^i_{1-\theta_i} = 1 - \Omega_i^+$ for all $i \in N$.

If $1 - \Omega_i^+ \geq \tau_{\delta,\lambda}^A$ for all $i \in N$, then $S^i_{1-\theta_i} \geq \tau_{\delta,\lambda}^A$ and, by Lemma 2, $BR_i = \theta_i = x_i^*$ for all $i \in N$. No agent has an incentive to deviate. This gives sufficiency.

Suppose there exists $i_0 \in N$ such that $1 - \Omega_{i_0}^+ < \tau_{\delta,\lambda}^A$, that is, $S^{i_0}_{1-\theta_{i_0}} < \tau_{\delta,\lambda}^A$. By Lemma 2, $BR_{i_0} = 1 - \theta_{i_0} \neq x_{i_0}^*$. Agent i_0 prefers to depart from her type because the neighbours who act differently from her are insufficient, in weighted terms, to justify choosing θ_{i_0}. Therefore, $x^* = \bar{\theta}$ is not a Nash equilibrium.

Consequently, $x^* = \bar{\theta}$ is a Nash equilibrium if and only if condition (10) is satisfied for every $i \in N$. This establishes necessity. □

Condition (10) captures the logic proper to anticoordination, which is structurally the mirror image of the coordination case. The sustainability of the fully satisfactory equilibrium does not require that every agent have many neighbours of her own type; on the contrary, it requires that neighbours of a *different* type be sufficiently influential, in the sense that their accumulated weight exceeds $\tau_{\delta,\lambda}^A$. An agent excessively surrounded by like-minded peers, that is, with $\Omega_i^+ > 1 - \tau_{\delta,\lambda}^A$, will not find sufficient differentiation pressure in her neighbourhood and will prefer to deviate toward the action opposite her type. As in the coordination case, it is the weighted influence of the relevant neighbours, not their number, that determines sustainability, and by (8) the direction of the departure from a simple count is again governed by the connectivity of the neighbours concerned. Therefore, a few well-connected dissenters can sustain the equilibrium where a numerical majority of peripheral ones cannot. Note that the same configuration is read in opposite ways by the two regions. An agent whose like-minded neighbours are unusually well connected has a high affinity weight, which protects her preferred action in coordination games and jeopardizes it in anticoordination ones.

Analogously, in the anticoordination area and following the results of Figure 3, node 4 has an incentive to deviate from her preference given that agents who act differently are insufficient. Condition (10) fails to hold since $\Omega_4^+ < \tau_{\delta\lambda}^A$.

3.3 Existence of Nash equilibrium

The analysis so far has asked whether particular action profiles, singled out by their economic content, are sustainable as equilibria, and has answered in terms of conditions on the primitives $(G, \bar{\theta})$ that can be checked by inspection. A different and logically independent question is whether the game possesses any equilibrium at all. The two questions do not reduce to one another since an existence result identifies some profile without describing it, while Propositions 4–6 describe named profiles without guaranteeing that the equilibrium set is non-empty when those profiles fail. We now address the second question. The results of the previous subsection settle the existence question in two of the three regions but leave it open in the third. In coordination games, Proposition 4 shows that both homogeneous profiles

are Nash equilibria for every network and every preference profile, so a pure-strategy equilibrium always exists in $\mathcal{R}_C$ whatever the topology and whatever the distribution of types. In dominant-action games, Lemma 3 shows that θ_i is a dominant action for every agent, so $x^* = \bar{\theta}$ is the unique equilibrium of every game in $\mathcal{R}_D$. Anticoordination games admit neither argument. Part (2) of Proposition 4 establishes that no homogeneous profile survives in $\mathcal{R}_A$, and Proposition 6 shows that the fully satisfactory profile survives only when every agent's affinity weight is low enough; since $\tau^A_{\delta,\lambda} > 0$ throughout $\mathcal{R}_A$, an agent all of whose neighbours share her type has $\Omega_i^+ = 1$ and violates condition (10) whatever the values of δ and λ. Both of the profiles studied above may therefore fail simultaneously, and nothing established so far guarantees that any other profile takes their place.

The difficulty is not merely that our two candidates have been exhausted. Order the action profiles by the set of agents choosing action 1. In $\mathcal{R}_C$ the best-response correspondence is monotone with respect to this order: by Lemma 1, an increase in the weight of neighbours choosing a given action makes that action weakly more attractive, so the game exhibits strategic complementarities and a fixed point follows from standard lattice arguments. In $\mathcal{R}_A$ the correspondence is antitone, since by Lemma 2 the same increase makes the action less attractive. Applying a lattice argument to the correspondence composed with itself then yields a pair of profiles that map into each other rather than a fixed point, which is the structure of a best-response cycle and not of an equilibrium. Anticoordination games on networks can indeed fail to possess pure-strategy equilibria, so the question is substantive.

We resolve it by replacing the search for an equilibrium by the search for a maximum. Suppose there existed a single real-valued function P defined on action profiles with the property that, whenever an agent contemplates changing her action, the resulting change in her own utility and the resulting change in P always have the same sign. Any profile at which P cannot be increased by altering one coordinate would then be a profile at which no agent can increase her utility by altering her own action, that is, a Nash equilibrium; and since the set of profiles is finite, such a maximum necessarily exists. A game admitting a function of this kind is a *potential game*, a class introduced by Monderer & Shapley (1996) and used in the analysis of network games by, among others, Bramoullé et al. (2014). In its exact form the requirement is that the change in P equal the change in the deviating agent's utility, which in our setting is too strong: it would force the two endpoints of a link to value that link identically, whereas $\omega_j^i \neq \omega_i^j$ in general, since agent i weighs j by j's connectivity relative to i's neighbourhood and agent j weighs i by i's connectivity relative to hers. What we establish instead is the weaker property, also due to Monderer & Shapley (1996), in which each agent's change in utility equals the change in P multiplied by a constant $w_i > 0$ that may vary across agents but not across profiles. Since the constants are strictly positive they do not affect any comparison, so the argument of the previous paragraph applies unchanged; each agent may be thought of as reading a common function on her own scale.

Theorem 1 (Weighted potential)**.** *Let $(G, \bar{\theta})$ be an arbitrary network and let $(\delta, \lambda) \in (0,1) \times (0,\infty)$. Define*

$$P(\mathbf{x}) \;=\; (2\delta - 1)(2 + \lambda) \sum_{\{i,j\}\,:\, j \in N_i} f_i f_j \, \mathbf{1}_{\{x_i = x_j\}} \;+\; \lambda \sum_{i \in N} f_i F_i \, \mathbf{1}_{\{x_i = \theta_i\}}. \tag{11}$$

Then for every agent $i \in N$, every pair of actions $x_i, x_i' \in \{0,1\}$ and every profile $\mathbf{x}_{-i}$,

$$u_i(x_i, \mathbf{x}_{-i}) - u_i(x_i', \mathbf{x}_{-i}) \;=\; w_i\big[P(x_i, \mathbf{x}_{-i}) - P(x_i', \mathbf{x}_{-i})\big], \qquad w_i \;=\; \frac{1}{2f_iF_i} \;>\; 0.$$

Proof. For an action profile $\mathbf{x}$ let

$$\sigma_i(\mathbf{x}) \;=\; \sum_{j \in N_i} \omega_j^i \, \mathbf{1}_{\{x_i = x_j\}}$$

be the accumulated weight of the neighbours whose action coincides with i's own, and define the auxiliary payoff

$$\hat{u}_i(\mathbf{x}) \;=\; (2\delta - 1)(2 + \lambda)\, \sigma_i(\mathbf{x}) \;+\; \lambda\, \mathbf{1}_{\{x_i = \theta_i\}}. \tag{12}$$

The proof proceeds in two steps. First we show that $\hat{u}_i$ generates the same incentives as u_i, and then that $\hat{u}_i$ is generated by the single function P.

If $x_i = \theta_i$ then $\sigma_i = S^i_{\theta_i}$, while if $x_i = 1 - \theta_i$ then $\sigma_i = S^i_{1-\theta_i} = 1 - S^i_{\theta_i}$. Substituting in (12),

$$\hat{u}_i(\theta_i, \mathbf{x}_{-i}) - \hat{u}_i(1-\theta_i, \mathbf{x}_{-i}) \;=\; (2\delta-1)(2+\lambda)\big[S^i_{\theta_i} - (1 - S^i_{\theta_i})\big] + \lambda \;=\; (2\delta-1)(2+\lambda)\big(2S^i_{\theta_i} - 1\big) + \lambda.$$

Halving this expression and grouping the constant terms,

$$\tfrac{1}{2}\big[\hat{u}_i(\theta_i, \mathbf{x}_{-i}) - \hat{u}_i(1-\theta_i, \mathbf{x}_{-i})\big] \;=\; (2\delta - 1)(2 + \lambda)\, S^i_{\theta_i} \;-\; \frac{(2\delta - 1)(2 + \lambda) - \lambda}{2},$$

and since

$$(2\delta - 1)(2 + \lambda) - \lambda \;=\; 2\delta(2 + \lambda) - (2 + \lambda) - \lambda \;=\; 2\delta(2 + \lambda) - 2 - 2\lambda \;=\; 2\big[\delta(2 + \lambda) - (1 + \lambda)\big],$$

we obtain

$$\tfrac{1}{2}\big[\hat{u}_i(\theta_i, \mathbf{x}_{-i}) - \hat{u}_i(1-\theta_i, \mathbf{x}_{-i})\big] \;=\; (2\delta - 1)(2 + \lambda)\, S^i_{\theta_i} \;-\; \big[\delta(2 + \lambda) - (1 + \lambda)\big],$$

which is precisely the difference $u_i(\theta_i, \mathbf{x}_{-i}) - u_i(1-\theta_i, \mathbf{x}_{-i})$ obtained in the proof of Lemma 1. Because the action set has two elements, any unilateral deviation is either from θ_i to $1 - \theta_i$ or the reverse, so

$$u_i(x_i, \mathbf{x}_{-i}) - u_i(x_i', \mathbf{x}_{-i}) \;=\; \tfrac{1}{2}\big[\hat{u}_i(x_i, \mathbf{x}_{-i}) - \hat{u}_i(x_i', \mathbf{x}_{-i})\big] \tag{13}$$

for all $x_i, x_i' \in \{0,1\}$ and all $\mathbf{x}_{-i}$.

Let us now to check that $\hat{u}_i$ is generated by the single function P. Collect the terms of $P(\mathbf{x})$ that involve x_i. A pair $\{i, j\}$ contributes to the first sum of (11) exactly when $j \in N_i$, and agent i contributes to the second sum only through her own indicator, so these terms are

$$(2\delta - 1)(2 + \lambda) \sum_{j \in N_i} f_i f_j \, \mathbf{1}_{\{x_i = x_j\}} \;+\; \lambda f_i F_i \, \mathbf{1}_{\{x_i = \theta_i\}}.$$

Factoring $f_i F_i$ and using $f_j/F_i = \omega_j^i$ from (4),

$$f_i F_i \Big[(2\delta - 1)(2 + \lambda) \sum_{j \in N_i} \omega_j^i \, \mathbf{1}_{\{x_i = x_j\}} \; + \; \lambda \, \mathbf{1}_{\{x_i = \theta_i\}} \Big] \;=\; f_i F_i \, \hat{u}_i(\mathbf{x}).$$

Every remaining term of P is unaffected by a change in x_i, so

$$P(x_i, \mathbf{x}_{-i}) - P(x_i', \mathbf{x}_{-i}) \;=\; f_i F_i \big[\hat{u}_i(x_i, \mathbf{x}_{-i}) - \hat{u}_i(x_i', \mathbf{x}_{-i}) \big].$$

Combining with (13) gives $P(x_i, \mathbf{x}_{-i}) - P(x_i', \mathbf{x}_{-i}) = 2 f_i F_i \big[u_i(x_i, \mathbf{x}_{-i}) - u_i(x_i', \mathbf{x}_{-i}) \big]$, which is the statement with $w_i = (2 f_i F_i)^{-1}$. □

Corollary 7 (Existence)**.** *For every network $(G, \bar{\theta})$ and every $(\delta, \lambda) \in (0,1) \times (0, \infty)$, the game possesses at least one Nash equilibrium in pure strategies. In particular, every anticoordination game has a pure-strategy equilibrium.*

Proof. The set of action profiles $\{0,1\}^N$ is finite, so P attains its maximum at some profile $\mathbf{x}^*$. Fix an agent i and an action $x_i' \in \{0,1\}$. By maximality, $P(\mathbf{x}^*) \geq P(x_i', \mathbf{x}_{-i}^*)$, and since $w_i > 0$, Theorem 1 gives $u_i(\mathbf{x}^*) \geq u_i(x_i', \mathbf{x}_{-i}^*)$. As i and x_i' were arbitrary, no agent has a profitable unilateral deviation from $\mathbf{x}^*$. □

Theorem 1 states that the agents' individual incentives, heterogeneous as they are, are all generated by a single function of the action profile, each agent reading it at her own scale w_i. Why a common function should exist at all is worth making explicit, because the reason lies in the weights rather than in the payoffs.

Corollary 7 therefore closes the gap left open in $\mathcal{R}_A$ and it also tells us what the equilibria of an anticoordination game look like when the two polar profiles are unavailable. They are neither homogeneous, since Proposition 4 excludes those, nor fully satisfactory when condition (10) fails; they are mixed configurations in which some agents act in accordance with their type while others act against it, and the identity of the second group is determined by the weighted topology of the network rather than by the preference profile alone.

The role played by the weights in this result should not be overstated, and neither should it be understated. Conditions (9) and (10), and indeed every result up to this point, use only the normalization (3) and would continue to hold if the weights ω_j^i were replaced by any other non-negative numbers summing to one within each neighbourhood. Theorem 1 is the first result that uses more, namely the attribute structure (4), and the following example shows that this additional structure cannot be dispensed with: a normalized but otherwise arbitrary system of weights may leave an anticoordination game with no pure-strategy equilibrium at all.

Example 1 (Normalization alone does not suffice)**.** *Let G be the triangle on $N = \{1,2,3\}$, let the three agents share the same preference, $\theta_1 = \theta_2 = \theta_3 = 0$, and replace the weights of the model by the cyclic system*

$$\omega_2^1 = \omega_3^2 = \omega_1^3 = a, \qquad \omega_3^1 = \omega_1^2 = \omega_2^3 = 1 - a, \qquad a \in (0,1),$$

illustrated in Figure 6. These weights satisfy the normalization (3) but are not of the form (4) for $a \neq \frac{1}{2}$: were they, the three ratios ω_2^1/ω_3^1, ω_3^2/ω_1^2 and ω_1^3/ω_2^3 would equal f_2/f_3, f_3/f_1

and f_1/f_2 respectively and would therefore multiply to one, whereas here they multiply to $a^3/(1-a)^3$. For $(\delta,\lambda) \in \mathcal{R}_A$, the game so defined possesses a Nash equilibrium in pure strategies if and only if

$$\min\{a,\, 1-a\} \;\geq\; \tau^A_{\delta,\lambda}. \tag{14}$$

In particular, taking $\delta = \frac{1}{3}$ and $\lambda = \frac{1}{2}$, which satisfy $\delta < \frac{1}{2+\lambda} = 0.4$ and give $\tau_{\delta,\lambda} = 0.8$ and $\tau^A_{\delta,\lambda} = 0.2$, the choice $a = 0.9$ leaves the game with no Nash equilibrium in pure strategies.

Proof. Since all three preferences equal 0, an action profile is described by the set $B \subseteq N$ of agents choosing action 1, that is, of agents acting against their type. For any agent i, the accumulated weight of the neighbours acting against i's preference is $S^i_{1-\theta_i} = \omega^i(B) := \sum_{j \in N_i \cap B} \omega^i_j$. By Lemma 2, the profile is a Nash equilibrium if and only if

$$\omega^i(B) \;<\; \tau^A_{\delta,\lambda} \;\text{ for every } i \in B, \qquad \text{and} \qquad \omega^i(B) \;\geq\; \tau^A_{\delta,\lambda} \;\text{ for every } i \notin B.$$

Recall from the proof of Lemma 2 that $\tau^A_{\delta,\lambda} \in (0, \frac{1}{2})$ in $\mathcal{R}_A$. We examine the eight possible sets B.

If $B = \emptyset$, then $\omega^i(B) = 0 < \tau^A_{\delta,\lambda}$ for every agent, so every agent should belong to B, a contradiction. If $B = N$, then $\omega^i(B) = 1 \geq \tau^A_{\delta,\lambda}$ for every agent, so no agent should belong to B, again a contradiction.

Suppose $|B| = 2$, say $B = \{1,2\}$. Agent 1 belongs to B and her only neighbour in B is agent 2, so the requirement is $\omega^1_2 = a < \tau^A_{\delta,\lambda}$; agent 2 belongs to B and her only neighbour in B is agent 1, so the requirement is $\omega^2_1 = 1 - a < \tau^A_{\delta,\lambda}$. Adding the two inequalities gives $1 < 2\tau^A_{\delta,\lambda}$, which contradicts $\tau^A_{\delta,\lambda} < \frac{1}{2}$. The two remaining pairs are obtained by the cyclic relabelling $1 \mapsto 2 \mapsto 3 \mapsto 1$, which leaves the weight system unchanged, so the same contradiction applies.

Suppose finally $|B| = 1$, say $B = \{1\}$. Agent 1 has no neighbour in B, so $\omega^1(B) = 0 < \tau^A_{\delta,\lambda}$ and her requirement holds. Agent 2 does not belong to B and her only neighbour in B is agent 1, so her requirement is $\omega^2_1 = 1 - a \geq \tau^A_{\delta,\lambda}$; agent 3 does not belong to B and her only neighbour in B is agent 1, so her requirement is $\omega^3_1 = a \geq \tau^A_{\delta,\lambda}$. Both hold if and only if $\min\{a, 1-a\} \geq \tau^A_{\delta,\lambda}$, and the cyclic symmetry gives the same condition for $B = \{2\}$ and $B = \{3\}$.

Hence a pure-strategy equilibrium exists if and only if (14) holds. For $a = 0.9$ and $\tau^A_{\delta,\lambda} = 0.2$ we have $\min\{0.9,\, 0.1\} = 0.1 < 0.2$, so no profile among the eight is an equilibrium. □

Two features of Example 1 deserve comment. The first is that the three agents share the same preference, so the failure of existence owes nothing to any conflict between types and everything to the asymmetry with which attention is allocated: agent 1 attends mostly to agent 2, who attends mostly to agent 3, who attends mostly to agent 1, and this circular pattern of influence admits no consistent resolution. The second is that such a failure can occur only in anticoordination games. Part (1) of Proposition 4 uses nothing beyond the normalization (3), so in $\mathcal{R}_C$ the two homogeneous profiles remain equilibria for any normalized weight system whatsoever, and in $\mathcal{R}_D$ the dominance of Lemma 3 is likewise independent of the weights. It is only where agents seek to differentiate themselves that the structure of the weights determines whether the game has an equilibrium, and it is there that Theorem 1 does its work.

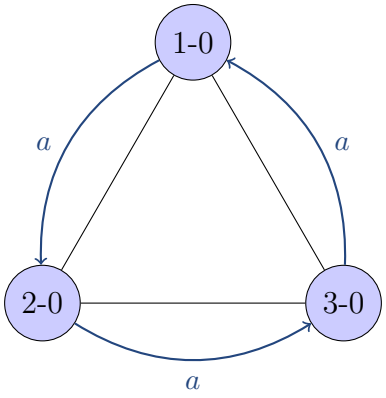


Figure 6: The weight system of Example 1. The first value numbers the node and the second shows its preference, which is the same for the three agents. Each arrow indicates the direction in which weight a is assigned: agent 1 assigns a to agent 2, agent 2 assigns a to agent 3, and agent 3 assigns a to agent 1; the reciprocal weights are $1-a$ in each case. The weights are normalized, but the cyclic asymmetry cannot be generated by any assignment of attributes to the nodes.

4 Discussion

This paper has introduced connectivity-dependent weights into a network game with idiosyncratic binary preferences. The two parts of the analysis answer complementary questions. The characterization of homogeneous and fully satisfactory equilibria determines which networks and preference profiles sustain configurations of independent economic interest, and requires only that each agent's attention be normalized; the potential delivers existence in full generality, including in the anticoordination region where both of those configurations may fail, but requires the connectivity structure of the weights and identifies no equilibrium in particular. Neither result subsumes the other, and Example 1 shows that the hypotheses genuinely differ.

The potential admits a reading that connects the model to the study of interacting particle systems. Writing $s_i = 2x_i - 1$ and $t_i = 2\theta_i - 1$ for the sign variables associated with actions and preferences, and using $\mathbf{1}_{\{x_i = x_j\}} = (1 + s_i s_j)/2$ together with $\mathbf{1}_{\{x_i = \theta_i\}} = (1 + s_i t_i)/2$, expression (11) becomes, up to an additive constant,

$$P \;=\; \frac{(2\delta - 1)(2 + \lambda)}{2} \sum_{\{i,j\}\,:\,j \in N_i} f_i f_j \, s_i s_j \;+\; \frac{\lambda}{2} \sum_{i \in N} f_i F_i \, t_i \, s_i .$$

The interaction between two neighbours is thus governed by the product of their net connectivities, and each agent is subject to an individual term whose sign is that of her preference and whose magnitude is her own connectivity weighted by that of her neighbourhood. The interaction is aligning in coordination games and opposing in anticoordination games, according to the sign of $2\delta - 1$. The mapping is not a contribution of the paper, but it locates the model within a familiar class and suggests that the tools developed for such systems, in particular for the analysis of ground states on heterogeneous graphs, may be informative about the structure of the equilibrium set in large networks.

Three questions arise directly from the results. The first concerns the class of weight systems for which the analysis goes through. Theorem 1 requires that the weight assigned to a neighbour depend on that neighbour alone, and Example 1 shows that mere normalization is insufficient; what remains open is a characterization of the intermediate systems, that is, of the precise condition on a normalized matrix of weights under which the associated game admits a potential. The second concerns selection. Where several equilibria coexist, as they do throughout the coordination region, the potential orders them, and the profiles it ranks highest weight each agent by her own connectivity and that of her neighbourhood rather than counting agents equally; whether this ordering has normative content, and how it relates to the welfare ranking of the same profiles, is not addressed here. The third concerns welfare itself. The relation between the fully satisfactory profile and the uniform conventions is not settled by our conditions: both may be equilibria of the same game, and which of them the agents would collectively prefer is an open question whose answer appears to depend on the region of the parameter space.

Two extensions of the model would broaden its scope. The analysis has been conducted under complete information, and Hernández et al. (2013) and Galeotti et al. (2010) show that moving to incomplete information, where agents know their own degree and type but not the full network, substantially reduces the set of equilibria in the unweighted case. Whether the affinity weight retains its role as a sufficient statistic when computed in expectation over an unknown neighbourhood is an open question, and one whose resolution would bring the model closer to the large-network settings in which weighted influence is most empirically relevant. Separately, the preference profile has been treated as exogenous. Allowing θ_i to respond to the network environment, in the spirit of Ghiglino & Tabasso (2024), and asking how the affinity weight shapes the resulting adjustment of preferences, would connect the static analysis developed here to the literature on norm formation.

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