Efficiency-Inducing Policy for Polluting Oligopolists *

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Abstract

This paper characterizes an efficiency-inducing policy for a polluting oligopoly when pollution abatement is technologically feasible, there are spillovers coming from the abatement effort of the firms, and environmental damages depend on the pollution stock. Using a dynamic policy game between the regulator and the oligopolists, we show that a tax-subsidy scheme can implement the efficient outcome as a regulated market equilibrium. The scheme consists of a tax on production and a subsidy that can be either on abatement effort or on abatement costs. Both schemes prescribe a different tax rule, but both implement the efficient outcome. If firms act strategically taking into account the evolution of the pollution stock when they decide on abatement and production, they will internalize the positive externality associated with the spillovers, and then the subsidy exclusively reflects the divergence between the social and private valuation of the pollution stock associated with the abatement decision. Consequently, the tax has to correct the two market failures associated with production, the market power of firms and the negative externality caused by pollution. Using a LQ (differential) policy game we show that the tax increases with the pollution stock for both schemes, and that the application of a subsidy on abatement costs leads to a laxer tax rule. Interestingly, it also yields a lower fiscal deficit at the steady state. Thus, from a fiscal perspective the policy recommendation is the application of a subsidy on abatement costs.

Keywords: oligopoly, homogeneous good, Cournot competition, abatement, spillovers, production tax, abatement subsidies, stock pollutant, differential games

JEL Classification System: H23, L12, L51, Q52, Q55

1 Introduction

In a very influential and seminal paper by Benchekroun and Long (1998) it is shown that there is a time-independent tax rule for polluting oligopolists that implements the efficient allocation as a regulated market equilibrium. The optimal tax increases with the pollution stock, but the authors found that it may be negative when the pollution stock is low, i.e. the optimal policy could consist of subsidizing production for an initial time interval.[1] This result is not so surprising if we consider that a polluting oligopoly is inefficient because two market failures are operating at the same time but with a different bias. On one hand, firms have *market power* that causes a reduction of production below the efficient level. On the other hand, pollution is an example of a *negative externality* that tends to increase production above the efficient level. In the first case, the optimal policy consists of a subsidy on production to close the gap between the price and the marginal revenue of the firms. In the second case, the optimal policy is to apply a tax on emissions to drive firms to internalize the negative externality. If the first market distortion dominates the second one, the optimal policy for polluting oligopolists would be a subsidy on production. Nevertheless, Martín-Herrán and Rubio (2021) have shown that if environmental damages are high enough the optimal policy consists of taxing production for any level of the pollution stock.

The aim of this paper is to characterize the efficiency inducing policy for polluting oligopolists if pollution abatement is technologically feasible. In this case, we have to distinguish between gross emissions linked to production and net emissions that depend on abatement effort developed by the firms. In this framework, a tax on net emissions cannot implement the efficient solution because it penalizes production and rewards abatement at the same rate, and we would need to do it at different rates to adjust the two control variables of the model, abatement and production. Recently, Martín-Herrán and Rubio (2018a) have addressed this issue for the case of a polluting monopoly. Following the argument we have previously presented, these authors show that the regulated market equilibrium is efficient for a policy mix that combines a tax on emissions with a subsidy on production. But still in this case the tax could be negative for low values of the pollution stock.[2] Although a subsidy on production in this framework is a policy to recover the efficiency of the market, it could be seen by the regulatory agencies as a

[1]In their model a unit of production generates one unit of emissions and there is no abatement. Thus, the tax on emissions operates as a tax on production.

[2]Borrero (2022) shows that this is a particular result that only happens for the monopoly. He proves that when the number of firms is higher than or equal to two, the first-best emission tax is always positive for any level of the pollution stock. Thus, in this case the tax would correct the externality and the subsidy would correct the market power of the firms.

policy against competition and it could be questioned and difficult to apply. To avoid this criticism, in this paper we propose, following Pal and Saha (2013), a policy that consists of penalizing production and rewarding abatement but a different rate i.e. using two different instruments.[3] In fact, we propose two tax-subsidy schemes that can implement the efficient allocation. A first policy mix that combines a tax on production with a subsidy on abatement effort, and a second policy mix for which the subsidy is on abatement costs.[4] Our model that can be read as either an extension of Martín-Herrán and Rubio (2018a) to the case of a polluting oligopoly or an extension of Benchekroun and Long (1998) to incorporate an abatement technology, also considers the possibility of spillovers in emission abatement. As spillovers generate a *positive externality* we address in this paper an interesting environmental regulation problem since we have a case in which there are three market failures.

For a general version of the model, we find that if firms act strategically taking into account the effects of their decisions on abatement and production on the evolution of the stock, because they expect that environmental policy will depend on the pollution stock level, they will *internalize* the spillovers associated with their abatement effort. Notice that the evolution of the pollution stock depends on aggregate emissions so that if firms care about the level of aggregate emissions, because they care about the evolution of the pollution stock, they will take into account the effect of their decisions on aggregate emissions, and when they decide on their abatement levels, they will also consider the effects of their decisions on other firms' emissions. This internalization by firms of spillovers will

[3]Pal and Saha (2013) show in a static model of a mixed duopoly with pollution that the government can implement the socially optimal outcome applying a tax on production and a subsidy on the abatement effort and keeping the public firm fully public.

[4]A classical paper on environmental regulation that incorporates a subsidy on costs is Katsoulacos and Xepapadeas (1996). In this paper, the authors analyze in a static setting the efficiency-inducing policy for a duopoly with spillovers consisting on a tax on emissions and a subsidy on R&D investment costs, where R&D investment reduces the emissions to output ratio. More recently, Saltari and Travaglini (2011) and Menezes and Pereira (2017) study the environmental regulation with subsidies in a dynamic setting. Saltari and Travaglini (2011) analyze a policy mix consisting on a tax on a polluting input and a subsidy on abatement investment, whereas Menezes and Pereira (2017) focus on a tax-subsidy scheme based on a tax on emissions and a subsidy on investment costs.

affect the optimal policy adopted by the regulator. On one hand, the subsidy will depend solely on the divergence between the social and private valuation of the pollution stock. When the subsidy is applied on abatement effort, the subsidy is proportional to the difference between the social and private shadow price of the pollution stock. This difference is multiplied by the effect that the abatement by one firm has on total abatement. If the subsidy is on abatement costs it is just equal to the ratio between the private and social shadow prices. In both cases, the subsidy will be positive. On the other hand, the tax, as in Benchekroun and Long (1998), has to correct the two market failures associated with production. Thus, we obtain that the optimal tax is equal to the difference between the marginal revenue and the price, that is negative, plus the difference between the social and private shadow prices, that is positive. The net effect could be negative. In any case, it is straightforward that the production tax rate will be lower than the abatement subsidy rate when this is applied to the abatement effort.

To advance in the analysis of the optimal policy rules, we solve, in the second part of the paper, a LQ (differential) policy game between the regulator and the oligopolists. The results confirm that, regardless of whether a subsidy is applied on abatement effort or on abatement costs, the tax could be negative for low levels of the pollution stock. Nevertheless, the tax increases with the pollution stock. A numerical exercise allows us to evaluate how the different parameters of the model influence the optimal tax rule and its steady-state value. We find that with high environmental damages, high spillovers, high efficiency of the abatement technology, and more competition in the market, we should expect a tax that is always positive. On the other hand, the subsidies are always positive and increase with the pollution stock. Interestingly, we find that although the differential game is linear-quadratic the subsidy rule when the subsidy is on abatement costs is not linear. Moreover, we would like to highlight that the model predicts that competition is good for the environment. We show analytically that with more firms in the industry, the steady-state pollution stock decreases, and the numerical exercise shows that although each firm's abatement and production decrease, the total production and abatement of the industry increase. Thus, total abatement is monotonically increasing with competition, and this increase is enough as to yield decreasing total emissions compatible with an increasing total output. Finally, we compare the optimal tax rules that are obtained when the two different subsidies are applied. From this comparison we find that the optimal tax rule is laxer when a subsidy on abatement costs is applied. In this case, both the intersection point with the vertical axis and the slope of the tax rule are lower and consequently the steady-state tax will be lower. Notice that both policy mix implement the same efficient solution. Thus, if the tax rule is less strict, the tax at the steady state will be lower. This means that lower tax revenues will be collected by the government if the subsidy is on abatement costs. However, from a fiscal point of view what is important is the fiscal balance of the policy mix. Unfortunately, the complexity of the fiscal balance expression prevents of finding any analytical conclusion for the comparison of the fiscal balances. Nevertheless, we can indicate that the numerical exercise shows that both policies present a fiscal deficit, but that the fiscal deficit is lower when the government subsidizes the abatement costs. Then, we can conclude that when this type of subsidy is used both the fiscal revenues and subsidy expenses are lower and that the net effect is a lower fiscal balance. With all cautions we should take since this result is based on numerical simulations, if the criterion for selecting the type of subsidy to accompany the tax is to select the one that leads to the most favorable fiscal balance, the policy recommendation would be to opt for a subsidy on abatement costs.

1.1 Literature Review[5]

A first list of paper addressing the regulation of firms with market power in the context of stock dynamics includes Bergstrom et al. (1981), Karp and Livernois (1992) and Karp (1992) for the case of a non-renewable resource and Benchekroun and Long (1998, 2002), Stimming (1999), Feenstra et al. (2001) and Yanase (2009) for the case of polluting firms.[6] Bergstrom et al. (1981) show that there exists a continuum of tax/subsidy schedules on output that lead to a monopoly to extract efficiently a non-renewable resource. However,

[5] Some of the references that appear in this review have already been commented in Martín-Herrán and Rubio (2018a, 2018b).

[6] Xepapadeas (1992) and Kort (1996) could be included in this list, but in their papers the market power of the polluting firms is not clearly recognized.

as these taxes/subsidies are time-dependent, they are not in general subgame perfect. Karp and Livernois (1992) design a subgame-perfect tax rule that implements the efficient outcome for a monopoly, Karp (1992) extends this result to the case of an oligopoly that extracts a common property non-renewable resource, and Benchekroun and Long to the case of a polluting oligopoly.[7] The two papers by Stimming (1999) and Feenstra et al. (2001) studying the case of a duopoly assume that environmental damages depend on current emissions and focus on investment in an abatement technology. The environmental policy in these papers is given and the analysis assesses the effects of a stricter environmental policy comparing taxes vs. emission standards. After the papers by Benchekroun and Long (1998, 2002), Yanase (2009) is a first paper where the environmental policy is *endogenously* determined. The author examines a non-cooperative (differential) policy game between national governments in a model of international pollution control of a stock pollutant in which duopolists compete myopically in quantities in a third country with product differentiation, and expense resources in abatement activities. The comparison of the Markov perfect Nash equilibrium of the game for different policy instruments establishes that an emission tax produces more pollution and lower welfare than those generated by a standard. This author assumes an end-of-the-pipe abatement technology as the one used in this paper, but without considering the possibility of spillovers on the abatement effort.[8]

Other papers addressing the environmental regulation of polluting firms with market power in a dynamic context are Benchekroun and Chaudhuri (2011), Martín-Herrán and

[7]Benchekroun and Long (2002) focus on the case of a polluting monopoly. For this case, they show that tax rules are not unique. Im (2002) shows that for a monopoly extracting a non-renewable resource a constant ad valorem subsidy induces the monopoly to behave efficiently if the demand is isoelastic and the marginal costs of extraction are constant. Daubanes (2011) clarifies that this is one case of a family of paths of ad valorem taxes/subsidies that induce efficiency in the resource's extraction, and shows that some of the paths may be strict taxes.

[8]Recently, Yanase and Kamei (2022) study a two-country differential game model of transboundary pollution with international polluting oligopolies. The authors assume that governments use permits to regulate pollution. They compare autarky and bilateral free trade and conclude that free trade is better for the environment than autarky.

Rubio (2018a) and Dragone et al. (2022). Benchekroun and Chaudhuri (2011) show that the imposition of a tax on emissions that depends on the pollution stock can induce stable cartelization in a polluting oligopoly making undesirable the regulation of the market. Martín-Herrán and Rubio (2018a) show that a tax-subsidy scheme consisting of taxing emissions and subsidizing production implements the efficient outcome as a regulated market equilibrium for a polluting monopoly with an abatement technology of the type proposed by Yanase (2009). They also show that taxes and standards are equivalent in a second-best setting. In this paper we extend this model to the case of an oligopoly incorporating spillovers on the abatement effort, but focusing on different tax-subsidy schemes. Dragone et al. (2022) also study the case of a polluting oligopoly with spillovers on the abatement effort where the damages depend on the pollution stock and the total output of the industry. However, the authors consider a tax on firms' accumulated emissions and focus mainly on the effect of competition on the aggregate abatement.

Another set of papers that analyze investment in pollution abatement capital in different settings are Saltari and Travaglini (2011), Karp and Zhang (2012), Menezes and Pereira (2017) and Martín-Herrán and Rubio (2018b). Saltari and Travaglini (2011) assume that uncertainty over the dynamics of pollution stock affects firm investment decisions and study, for the case of a competitive firm, how a tax-subsidy scheme based on a tax on the polluting input and a subsidy on investment influences the firm's decisions on investment. However, in their model there is no connection between the use of the polluting input and the evolution of the pollution stock. Karp and Zhang (2012) compare emission taxes and standards when a regulator and a representative firm have asymmetric information about abatement costs, and all agents use Markov perfect decision rules. The firm can reduce future abatement costs through investment. For a linear-quadratic specification of the model and using numerical methods, they find that a tax has some advantage over a standard. Menezes and Pereira (2017) study the dynamic competition of a duopoly in supply schedules that can invest in an abatement technology. In their model damages are linear in the pollution stock and there are also technological spillovers. The focus is on the characterization of the optimal policy mix consisting on a tax on emissions and a subsidy on investment costs assuming that the regulator can commit for the entire temporal horizon and that firms' production, investment and abatement capital are given by their steady-state values when the regulator decides the optimal policy. Our paper differs from this work mainly in three aspects. Firstly, we do not assume that the regulator can commit for the entire temporal horizon, instead we look for the *feedback Stackelberg equilibrium* of the differential game played by the regulator and the oligopolists, i.e. the regulator maximizes net social welfare subject to best responses of the firms to the policy adopted by the regulator. Secondly, we assume quadratic environmental damages while Menezes and Pereira (2017) assume a linear damage function. Thirdly, we consider two tax-subsidy schemes that are based on a tax on production instead of a tax on emissions and we also consider a subsidy on the abatement effort. Finally, Martín-Herrán and Rubio (2018b) analyze the second-best emission tax for a polluting monopoly with abatement investments investigating the consequences for investment of two different damages structure, one linear and one quadratic in the pollution stock.

Recently, Bisceglia (2020) characterizes the efficiency-inducing tax rule imposed on output for an oligopoly that exploits a common productive resource, and Benchekroun at al. (2022) propose for an oligopoly that extracts a common non-renewable resource a novel tax scheme to implement the efficient outcome where the tax bill paid by a firm depends only on the current resource stock. Finally, Feichtinger et al. (2022) present a model of a polluting common renewable resource exploited by an oligopoly where firms can invest in an abatement technology. The authors show that if the demand is linear, the extraction costs are linear and the access is regulated as to induce the industry to harvest at the maximum sustainable yield, then there exists a tax on accumulated emissions of the firm at which aggregate emissions drop to zero. Taxation induces firms to invest in the abatement technology and eliminate emissions.[9]

The remainder of the paper is organized as follows. Section 2 presents the model and derives the efficient conditions. Section 3 characterizes the first-best policy mix distinguishing between the two tax-subsidy schemes studied in this paper. In Section 4 a

[9]In their model, if access to the common resource is limited to attain the maximum sustainable yield, the emission tax has no impact on the environmental damages.

LQ policy game is solved. Section 5 offers some concluding remarks and points out lines for future research.

2 The Model and the Efficient Conditions

We consider a Cournot oligopoly that faces a market demand represented by the decreasing inverse demand function $P(Q(t))$ where $Q(t) = \sum_{i=1}^{n} q_i(t)$ is the output of the industry at time t and $n \geq 2$ is the number of firms. Firms produce an homogeneous good using the same productive technology, described by the cost function $PC = cq_i(t)$. The production process generates pollution emissions, but after an appropriate choice of measurement units we can say that each unit of output generates one unit of pollution. However, emissions can be reduced without declining output if firms employ an abatement technology.[10] Following Poyago-Theotoky (2007), the abatement technology is assumed to be the end-of-the-pipe type.[11] For this type of abatement technology the emission function is $e_i(t) = q_i(t) - w_i(t) - \beta \sum_{j \neq i}^{n} w_j(t)$, where $w_i(t)$ is the abatement effort of firm i, and parameter $\beta \in [0, 1]$ accounts for the presence of spillovers in emission abatement. The abatement cost function is represented by $AC(w_i(t))$ with $(AC)'(w_i)$ and $(AC)''(w_i)$ positive. The focus of the paper is on a stock pollutant that evolves according to the following differential equation

$$\dot{x}(t) = \sum_{i=1}^{n} \left(q_i(t) - w_i(t) - \beta \sum_{j \neq i}^{n} w_j(t) \right) - \delta x(t), \quad x(0) = x_0 \geq 0, \tag{1}$$

where $x(t)$ stands for the pollution stock and $\delta > 0$ for the decay rate of pollution. The environmental damages are given by the function $D(x(t))$ that is assumed strictly convex. Thus, the policy game we analyze in this paper is a *differential game* between a welfare

[10]This model can be seen in a certain way, as already mentioned in the introduction, as an extension of the model studied in Martín-Herrán & Rubio (2018a) to an oligopolistic market with spillovers in emission abatement, instead of a monopoly. As such both models share certain important ingredients and features that are repeated here for completeness and readability.

[11]This approach has been adopted in a dynamic setting by other authors as Yanase (2009), Menezes and Pereira (2017), Martín-Herrán and Rubio (2018a) and Dragone et al. (2022).

maximizing regulator and profit maximizing oligopolists. Before analyzing it, we first derive the first-order conditions that characterize the efficient outcome.

The efficient conditions are obtained from the maximization of the discounted present value of net social welfare defined as the difference between gross consumer surplus minus costs and environmental damages.[12]

$$\max_{q_1,\ldots,q_n,w_1,\ldots,w_n} \int_0^\infty e^{-rt}\left\{\int_0^Q P(Q')dQ' - cQ - \sum_{i=1}^n AC(w_i) - D(x)\right\}dt$$

$$\text{s.t. } \dot{x} = \sum_{i=1}^n\left(q_i - w_i - \beta\sum_{j\neq i}^n w_j\right) - \delta x, \quad x(0) = x_0 \geq 0,$$

where r is the time discount rate.

Solving by dynamic programming, the solution to this dynamic optimization problem must satisfy the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} rW(x) = \max_{q_1,\ldots,q_n,w_1,\ldots,w_n} & \left\{\int_0^Q P(Q')dQ' - cQ - \sum_{i=1}^n AC(w_i) - D(x)\right. \\ & \left. +W'(x)\left(\sum_{i=1}^n\left(q_i - w_i - \beta\sum_{j\neq i}^n w_j\right) - \delta x\right)\right\}, \end{aligned} \quad (2)$$

where $W(x)$ represents the maximum discounted present value of net social welfare for the current value, x, of the pollution stock.

The maximization of the right-hand side (RHS) of the HJB equation yields the following first-order conditions (FOCs)

$$P = c - W'(x), \quad i = 1,...n, \quad (3)$$

$$(AC)'(w_i) = -W'(x)(1+\beta(n-1)), \quad i = 1,...,n. \quad (4)$$

The first FOC establishes that the price must be equal to the marginal costs that include the marginal cost of production plus the *social* valuation (shadow price) of the pollution stock. The latter is given by the reduction in the present value of the net social welfare because of an increase in the pollution stock caused by an increase in production. On the other hand, the second FOC requires that the marginal cost of abatement is equal

[12]Time argument will be eliminated when no confusion arises.

to the marginal benefit defined by the increase in the present value of the net social welfare caused by a reduction in the stock because of an increase in abatement that with spillovers is given by $1 + \beta(n-1)$. Notice that $W'(x)$ is a marginal cost when we are considering an increase in production, and it stands for a marginal benefit when we are evaluating an increase in abatement.

To implement these conditions as a regulated market equilibrium we propose two tax-subsidy schemes. The first scheme combines a tax on gross emissions, that in our model operates as a tax on production, with an abatement subsidy. The second scheme uses a subsidy on abatement cost instead of an abatement subsidy. In the next section, we calculate the *stagewise feedback Stackelberg equilibrium* (SFSE) of a (differential) policy game where the regulator who selects the level of the policy instruments is the leader and the firms that choose the levels of production and abatement the followers, and we show that using these schemes the regulated market equilibrium will be efficient.

3 The First-Best Policy

The SFSE is based on the assumption that the regulator moves first in each moment. To find the regulator's optimal policy, we apply backward induction, substituting the firms' reaction functions in the regulator's HJB equation, and computing the optimal strategy by maximizing the right-hand side of this equation. The resulting outcome is a stagewise feedback Stackelberg solution, which is a Markov-perfect equilibrium. For this kind of equilibria no commitment is required for the entire temporal horizon. For our model, this equilibrium is time consistent and also satisfies subgame perfection.[13]

[13] Martín-Herrán and Rubio (2021) showed that the SFSE coincides with the *Global Stackelberg Equilibrium* used by Benchekroun and Long (1998) if the focus is on the design of the first-best policy that implements the efficient outcome as a regulated market equilibrium.

3.1 Tax-Subsidy Scheme I

The output and abatement selection occurs in the second stage. Firm i chooses its output and abatement to maximize the discounted present value of net profits

$$\max_{q_i, w_i} \int_0^{\infty} e^{-rt} \left\{ P(Q)q_i - cq_i - AC(w_i) - \tau q_i + v w_i \right\} dt,$$

subject to differential equation (1) where τ is the production tax and v stands for a subsidy on abatement. We assume that the firm acts strategically at this stage because it is aware that the dynamics of the stock will be taken into account by the regulator to set up the optimal policy. This implies that firms will take into account not only the effect that its abatement effort has on its own emissions, but also the effect that its abatement has on the net emissions of the other firms in the industry. In other words, if the firms take into account the dynamic constraint when they decide on abatement and production, they will *internalize* the spillovers of their abatement effort, because the evolution of the stock depends on aggregate net emissions. In this case, the spillovers do not cause any market distortion and the regulator has to correct only the market distortions caused by the firms' market power and the negative externality.

The solution to this dynamic optimization problem must satisfy the following HJB equation

$$\begin{aligned} rV^I(x) &= \max_{q_i, w_i} \{ P(Q)q_i - cq_i - AC(w_i) - \tau q_i + v w_i \\ &+ (V^I)'(x) \left(\sum_{i=1}^{n} \left(q_i - w_i - \beta \sum_{j \neq i}^{n} w_j \right) - \delta x \right) \right\}, \end{aligned}$$

where $V^I(x)$ stands for the maximum discounted present value of net profits for the current value, x, of the pollution stock.[14]

From the FOCs for the maximization of the right-hand side of the HJB equation, we get

$$P' q_i + P = c + \tau - (V^I)'(x), \; i = 1, ..., n, \tag{5}$$

$$(AC)'(w_i) = v - (V^I)'(x)(1 + \beta(n-1)), \; i = 1, ..., n. \tag{6}$$

[14] The superscript I stands for tax-subsidy scheme I.

The left-hand side (LHS) of the first FOC stands for the marginal revenue of the firm and the RHS represents the marginal costs that include the marginal cost of production, the tax and the *private* valuation (shadow price) of the pollution stock. The latter is given by the reduction in the present value of the firm's net profits because of an increase in the pollution stock caused by an increase in production. On the other hand, the LHS of the second FOC represents the marginal cost of abatement and on the RHS appears the marginal benefits. These marginal benefits include the subsidy, and the increase in the present value of the firm's profits because of the reduction in the pollution stock that with spillovers is given by $V^I(x)(1+\beta(n-1))$. As we commented above, we can see that the spillovers are internalized by firms as long as they act strategically taking into account how their decisions on abatement and production will affect the evolution of the pollution stock. Notice that $(V^I)'(x)$ is a marginal cost when we are considering an increase in production, and it stands for a marginal benefit when we are evaluating an increase in abatement. The system of reaction functions (5) implicitly defines the firm's strategy $q_i(\tau, x)$ and (6) directly yields $w_i(v, x)$. Notice that given the structure of the FOCs, the optimal production does not depend on the subsidy and the optimal abatement effort does not depend on the tax.

In the first stage, the regulator selects the emission tax rate and subsidy by unit of abatement that maximize net social welfare defined as the sum of consumer surplus and monopoly net profits plus tax revenues minus subsidies and environmental damages:

$$\max_{\tau,v} \int_0^\infty e^{-rt} \left\{ \int_0^Q P(Q')dQ' - PQ + \sum_{i=1}^n \pi_i + \tau Q - v\Omega - D(x) \right\} dt,$$

subject to differential equation (1), where π_i stands for firm i's net profits and $\Omega = \sum_{i=1}^n w_i$. Notice that consumer expenses and firms' revenues on one hand, and firms' tax expenses and subsidies and regulator tax revenues and subsidy expenses on the other hand, cancel out. Therefore, this optimization problem can be rewritten as

$$\max_{\tau,v} \int_0^\infty e^{-rt} \left\{ \int_0^{Q(\tau,x)} P(Q')dQ' - cQ(\tau,x) - \sum_{i=1}^n AC(w_i(v,x)) - D(x) \right\} dt,$$

where $Q(\tau,x) = \sum_{i=1}^n q_i(\tau,x)$.

The solution to this dynamic optimization problem must satisfy the following HJB equation

$$\begin{aligned} rW(x) &= \max_{\tau,v} \left\{ \int_0^{Q(\tau,x)} P(Q')dQ' - cQ(\tau,x) - \sum_{i=1}^{n} AC(w_i(v,x)) - D(x) \right. \\ &+ \left. W'(x)\left(Q(\tau,x) - \sum_{i=1}^{n}\left(w_i(v,x) + \beta \sum_{j\neq i}^{n} w_j(v,x)\right) - \delta x\right)\right\}. \end{aligned} \quad (7)$$

From the FOCs for the maximization of the RHS of the HJB equation, we get

$$(P - c + W'(x))\frac{\partial Q}{\partial \tau} = 0, \quad (8)$$

$$-\sum_{i=1}^{n}((AC)'(w_i) + W'(x)(1+\beta(n-1)))\frac{\partial w_i}{\partial v} = 0. \quad (9)$$

Assuming that both output and abatement are affected by the tax and subsidy, it is immediate that these conditions are satisfied if the efficient conditions hold. Thus, using the efficient conditions along with FOCs (5) and (6) we can characterize the *first-best policy.* Conditions (3) and (5) allow us to define the optimal tax

$$\tau^{I^*}(x) = P'q_i - (W'(x) - (V^I)'(x)), \quad (10)$$

and conditions (4) and (6) the optimal subsidy

$$v^{I^*}(x) = -(W'(x) - (V^I)'(x))(1+\beta(n-1)). \quad (11)$$

Notice that in both cases the policy instrument reflects the difference between the social and private valuations of a variation in the pollution stock. In the case of the tax, of an increase in the pollution stock caused by an increase in net emissions provoked by an increase in output. In the case of the subsidy, of a decrease in the pollution stock explained by a decrease in net emissions consequence of an increase in the abatement effort. With spillovers an increase in the abatement effort by a firm implies a reduction in total emissions equal to $1+\beta(n-1)$, and hence, for this reason in (11) the difference between the social and private valuations is multiplied by this term. In the case of the tax, we find an additional term equal to the difference between the marginal revenue of the firms and the price that appears because the firms have market power. With the tax, the regulator is correcting two distortions in the market allocation, the market power of the firms and a negative externality. For this reason, the tax has two components. The first, that is negative, operates as a subsidy on production to correct the market power of the firms, closing the gap between the price and the marginal revenue. The second, that is expected to be positive, operates as a tax on emissions to correct the negative externality.[15] Thus, we can state that

Remark 1 *The production tax could be negative if the distortion cause by the market power of the firms is bigger than the distortion caused by the negative externality.*

Nevertheless, if the main problem in the market is pollution, we should expect the opposite result and the optimal policy would be to tax gross emissions. Observe that even with a subsidy as this is on abatement, still the tax has to correct the two distortions associated with the production as occurs in Benchekroun and Long (1998).

On the other hand, if we compare the optimal levels of the two instruments we obtain the following expression

$$\tau^{I*}(x) - v^{I*}(x) = P'q_i + (W'(x) - (V^I)'(x))\beta(n-1).$$

Then, if the social shadow price of pollution stock is higher than the private one, we can conclude that

Remark 2 *The production tax rate is lower than the abatement subsidy rate.*

The difference between the two rates is due to firms' market power and the existence of spillovers in the abatement effort. If the firms have no market power, i.e. if $P' = 0$, the difference is negative because of the spillovers. If the spillovers are zero the difference is negative because of the firms' market power. Obviously, if the firms are price-takers and there are not spillovers the two rates coincide and with a tax on emissions would be enough to implement the efficient outcome.

[15]In the LQ policy game we study in the next section we confirm that $|W'(x)| > \left|(V^I)'(x)\right|$ so that $-(W'(x) - (V^I)'(x))$ is positive. Notice that if this was not the case, the tax and the subsidy would be negative for all x.

3.2 Tax-Subsidy Scheme II

With a subsidy on abatement costs, firm i chooses its output and abatement to maximize the discounted present value of net profits given in this case by

$$\max_{q_i, w_i} \int_0^\infty e^{-rt} \left\{ P(Q)q_i - cq_i - (1-v)AC(w_i) - \tau q_i \right\} dt,$$

subject to differential equation (1) where τ again is the production tax and $v \in (0,1)$ stands for a subsidy on abatement cost. v represents the percentage of the abatement costs that are covered by the subsidy. With this scheme the FOC (5) does not change, but the FOC (6) will read[16]

$$(1-v)(AC)'(w_i) = -\left(V^{II}\right)'(x)(1+\beta(n-1)). \tag{12}$$

Then, using (4) we obtain that the optimal subsidy is given by the following expression

$$1 - v^{II*}(x) = \frac{\left(V^{II}\right)'(x)}{W'(x)}. \tag{13}$$

In this case, the subsidy is also given by the different valuation that firms give to the pollution stock but not as a difference between the social and private shadow prices of the pollution stock, as occurs when the subsidy is on abatement effort, but as a ratio, as a percentage of the private shadow price over the social shadow price. Notice that the subsidy will be positive only if $\left|\left(V^{II}\right)'(x)\right| < |W'(x)|$. This point is confirmed in the LQ policy game we analyze in the next section.

Thus, the efficient solution could be implemented as a regulated market equilibrium using these two tax-subsidy schemes. Consequently, we cannot rank them looking at the net social welfare that is achieved using the two tax-subsidy schemes because both implement the efficient solution, i.e. with both schemes the maximum net social welfare is achieved. An alternative would be to assess them from a fiscal perspective. The scheme to recommend would be the one that yields a higher/lower fiscal superavit/deficit. In the next section, we introduce a LQ policy game that allows us to advance in the analysis of these two tax-subsidy schemes.

[16]The superscript II stands for tax-subsidy scheme II.

4 The LQ Policy Game

The LQ differential game we analyze in this section considers a polluting oligopoly that faces a linear (inverse) demand function given by $P = a - Q$, where P is the price and Q the total output of the industry with $a > c$. On the other hand, we assume a quadratic abatement cost function given by $AC(w) = \gamma w^2/2$. The abatement technology has decreasing returns to scale, with the parameter γ measuring the extent of such decreasing returns. The disutility from environmental deterioration is given by the damage function $D(x) = dx^2/2,\ d > 0$. Next, we characterize the efficient solution.

4.1 The Efficient Solution

If we focus on the symmetric solution the optimal strategies for production and abatement from (3) and (4) are

$$q^*(x) = \frac{1}{n}(a - c + W'(x)), \tag{14}$$

$$w^*(x) = -\frac{1+\beta(n-1)}{\gamma}W'(x). \tag{15}$$

Then, optimal emissions can be obtained as the difference between optimal production (gross emissions) and abatement

$$\begin{aligned} e^*(x) &= q^*(x) - (1+\beta(n-1))w^*(x) \\ &= \frac{a-c}{n} + \frac{\gamma + n(1+\beta(n-1))^2}{n\gamma}W'(x). \end{aligned} \tag{16}$$

Now, substituting production and abatement by the efficient strategies (14) and (15) in the regulator's HJB equation (5) for the LQ policy game, and rearranging terms we obtain the following nonlinear differential equation

$$rW(x) = \frac{1}{2}(s^2 - dx^2) + (s - \delta x)W'(x) + \frac{\gamma + n(1+\beta(n-1))^2}{2\gamma}(W'(x))^2, \tag{17}$$

where $s = a - c > 0$.

In order to find the solution for this equation, we guess a quadratic representation for the value function W :

$$W(x) = \frac{A_r}{2}x^2 + B_r x + C_r,$$

which implies that $W'(x) = A_r x + B_r$ and where $A_r,\ B_r$ and C_r are unknowns to be determined.[17]

The substitution of $W(x)$ and $W'(x)$ into (17) gives a system of Riccati equations that must be satisfied for every x. Selecting the stable solution of this system, that requires that $d\dot{x}/dx < 0$, we obtain the following values for the first two coefficients of the regulator's value function

$$A_r = \frac{\gamma(r+2\delta) - (\gamma^2(r+2\delta)^2 + 4d\gamma g)^{1/2}}{2g} < 0, \tag{18}$$

$$B_r = \frac{s\gamma A_r}{\gamma(r+\delta) - gA_r} < 0, \tag{19}$$

where $g = \gamma + n(1+\beta(n-1))^2$. Then, the optimal strategies for production, abatement and emissions read

$$q^*(x) = \frac{s(\gamma(\delta+r) - n(1+\beta(n-1))^2 A_r)}{n(\gamma(\delta+r) - gA_r)} + \frac{A_r}{n}x, \tag{20}$$

$$w^*(x) = \frac{s(1+\beta(n-1))A_r}{\gamma(\delta+r) - gA_r} - \frac{1+\beta(n-1)}{\gamma}A_r x, \tag{21}$$

$$e^*(x) = \frac{\gamma s(r+\delta)}{n(\gamma(\delta+r) - gA_r)} + \frac{g}{n\gamma}A_r x. \tag{22}$$

From the optimal strategy for emissions and the differential equation (1), and taking into account the first Riccati equation for A_r, the steady-state pollution stock is obtained

$$x^{SS} = \frac{s\gamma(r+\delta)}{(\gamma + n(1+\beta(n-1))^2)d + \gamma\delta(r+\delta)}.$$

From the expression above, it can be easily shown that the steady-state pollution stock increases with s, r and γ, while decreases with n, d, and β. Thus, we find that more competition not only reduces the market power of the firms, but also reduces the long-run equilibrium pollution stock. Thus, in our model competition is good for the environment.

According to the optimal strategies, production and emissions decrease with the pollution stock whereas abatement is increasing. Thus, there exists a level for the pollution stock for which emissions are zero. From equation $e^*(x) = 0$ this value reads

$$x_e = -\frac{s\gamma^2(r+\delta)}{A_r g\,(\gamma(\delta+r) - gA_r)}. \tag{23}$$

[17] The subscript r refers to the regulator and stands for the efficient solution.

This threshold can be easily compared with the steady-state value of the pollution stock, as follows:

$$x_e - x^{SS} = -\frac{s\gamma^3\delta(r+\delta)}{A_r g\,(\gamma\delta - gA_r)\,(\gamma(\delta+r) - gA_r)}.$$

The difference above is positive because A_r is negative.

Proposition 1 *The efficient solutions for the total output, the abatement and the emissions are non-negative in the interval* $[0, x_e]$ *with the steady-state pollution stock* x^{SS} *belonging to this interval. In this interval, total output and emissions decrease and abatement increases with the pollution stock.*

Finally, we characterize the dynamics of the pollution stock. Substituting emissions given by (22) in the dynamics of the pollution stock defined by (1), we obtain the following differential equation for the pollution stock

$$\dot{x} = \frac{s\gamma(r+\delta)}{\gamma(r+\delta) - gA_r} + \left(\frac{gA_r}{\gamma} - \delta\right)x,$$

whose solution is

$$x^*(t) = (x_0 - x^{SS})e^{\alpha t} + x^{SS}, \text{ with } \alpha = \frac{gA_r}{\gamma} - \delta < 0, \tag{24}$$

for x_0 in the interval $[0, x_e^*]$. Then, the dynamics of the model can be summarized as follows

Remark 3 *If* x_0 *is lower than* x^{SS}*, abatement increases asymptotically to its steady-state value, whereas production and emissions decrease. However, if* x_0 *is larger than* x^{SS}*, the dynamics is the opposite and abatement decreases asymptotically to its steady-state value, whereas production and emissions increase.*

4.2 Tax-subsidy scheme I

Once the efficient solution has been obtained we calculate the optimal policy rules that implement the efficient outcome as a regulated market equilibrium. According to (10) the optimal tax for a linear demand function is ${\tau^I}^*(x) = -q^*(x) - (W'(x) - (V^I)'(x))$, that using (14) yields

$$ {\tau^I}^*(x) = (V^I)'(x) - \frac{s + (n+1)W'(x)}{n}. \quad (25) $$

Thus, in order to completely characterize the optimal tax, and also the optimal subsidy, we need to solve the firm's HJB equation. With this aim, we substitute the tax and the subsidy given by (25) and (11), and production and abatement defined by (14) and (15) in the firm's HJB equation

$$ rV^I(x) = (a - nq)q - cq - \tau q + vw - \frac{\gamma}{2}w^2 + (V^I)'(x)(n(q - (1 + \beta(n-1)w) - \delta x), $$

and we obtain the following differential equation

$$ \begin{aligned} rV^I(x) &= \frac{1}{n^2}(s + W'(x))^2 + \frac{(1+\beta(n-1))^2}{2\gamma}W'(x)^2 \\ &+ (n-1)\frac{\gamma + n(1+\beta(n-1))^2}{\gamma n}(V^I)'(x)W'(x) + \frac{(n-1)s - \delta n x}{n}(V^I)'(x). \end{aligned} \quad (26) $$

In order to solve this equation, we also guess a quadratic representation

$$ V^I(x) = \frac{A_f^I}{2}x^2 + B_f^I x + C_f^I, $$

that yields $(V^I)'(x) = A_f^I x + B_f^I$.[18] The substitution of $V^I(x)$ and $(V^I)'(x)$ along with $W'(x)$ into (26) gives a system of Riccati equations whose solution for coefficients A_f^I and B_f^I is

$$ A_f^I = \frac{hA_r^2}{n(n\gamma(r+2\delta) - 2(n-1)gA_r)} > 0, \quad (27) $$

$$ B_f^I = \frac{(n\gamma(r+2\delta)(2s\gamma + hB_r) + (n-1)(s\gamma(h-4g) - ghB_r)A_r)A_r}{n(n\gamma(r+\delta) - (n-1)gA_r)(n\gamma(r+2\delta) - 2(n-1)gA_r)}, \quad (28) $$

where $h = 2\gamma + n^2(1 + \beta(n-1))^2$ and $A_r < 0$ is given by (18). Then, eliminating $(V^I)'(x)$ and $W'(x)$ in (25) using the coefficients of the value functions, the optimal tax is obtained.

[18] The subscript f is used to represent the coefficients of the firm's value function and the superscript I denotes that the tax-subsidy scheme I is applied.

Proposition 2 *The optimal tax is given by the following rule*

$$\tau^{I*}(x) = -\frac{1}{n}(s + (n+1)B_r - nB_f^I) + \frac{(h + 2(n+1)(n-1)g)A_r^2 - (n+1)\gamma(r+2\delta)A_r}{n^2\gamma(r+2\delta) - 2(n-1)ngA_r}x, \tag{29}$$

where A_r is negative. The tax increases with the pollution stock, but it could be negative for low values of the pollution stock.

It is easy to find values for the parameters for which the intersection point with the vertical axis of the tax rule (29) is negative.[19] When this is the case the optimal policy consists of setting up a subsidy for low values of the pollution stock as in Benchekroun and Long (1998) and exactly for the same reasons. Although in our model, we consider that there are spillovers generated by the abatement effort, as the firms internalize the spillovers, since they take into account the evolution of the stock when they decide on abatement and production, the subsidy only corrects the divergence between the private and social valuation of a variation in the pollution stock caused by a variation of the abatement effort. Then, the tax, as expression (10) shows, must correct the market power of the firms and the negative externality caused by production. The result is that the sign of the optimal policy given by expression (10) remains undetermined. Nevertheless, Prop. 2 establishes that the sign of the policy given by (29) also depends on the pollution stock, and that regardless of whether the tax is negative or positive when $x = 0$, the tax increases with the pollution stock.

To obtain the optimal subsidy we only need to eliminate $(V^I)'(x)$ and $W'(x)$ in (11) using the coefficients of the value functions already computed.

Proposition 3 *The optimal subsidy is given by the following rule*

$$v^{I*}(x) = (1 + \beta(n-1))\left(B_f^I - B_r + \frac{(h + 2(n-1)ng)A_r^2 - n^2\gamma(r+2\delta)A_r}{n(n\gamma(r+2\delta) - 2(n-1)gA_r)}x\right), \tag{30}$$

[19]For instance for $s = 1000$, $\beta = 0.025$, $\gamma = 1.5$, $\delta = 0.01$, $d = 0.01$, $n = 2$ and $r = 0.03$, the intersection point with the vertical axis of the tax rule is negative. For the same values, except $\gamma = 1.25$ and $d = 0.025$, we have also a negative value for the intersection point with the vertical axis. However, with $\gamma = 1.5$ and $d = 0.025$ the optimal tax is positive for $x = 0$.

where A_r is negative. The subsidy increases with the pollution stock and it is positive for all $x \in [0, x_e]$.

Proof. See Appendix. ■

Unlike the tax, the subsidy cannot be negative. This result establishes according to expression (11) that the social shadow price of the pollution stock is larger than the private shadow price for all x, i.e. $|W'(x)| > \left|(V^I)'(x)\right|$. Then, we can confirm that the second term on the LHS of expression (10) is positive. Thus, the tax presents two components, one negative, equal to the difference between the marginal revenue and the price, and one positive, equal to the difference between the social shadow price of the pollution stock and its private valuation.

4.2.1 Numerical example and sensitivity analysis

In Subsection 4.1 we have analyzed the effects of the model parameters on the steady-state pollution stock. However, it is difficult to do the same exercise for the other variables of the model, because when a parameter changes not only the pollution stock is affected, but also the optimal policy rules obtained in the previous subsection. Thus, to know the effect of a change in one parameter on the variables of the model, we need to evaluate how the steady-state pollution stock is affected, but also how the change affects the slope and the intersection point with the vertical axis of the optimal policy rules. The same occurs for the optimal strategies for production, abatement and emissions. The complexity of the expressions prevents of obtaining analytical results. For this reason, we present a numerical exercise to get an intuition.

Let consider the following values of the parameter as a baseline case:

$$a - c = 1000,\ \delta = 0.01,\ r = 0.03,\ n = 2,\ \beta = 0.025,\ \gamma = 3,\ d = 0.025. \qquad (31)$$

From this baseline case, we carry out a sensitivity analysis with respect to the following parameters: environmental damage (d), abatement spillover (β), abatement efficiency (γ) and degree of industry competition (n). For each parameter we consider five different values. In each table (Tables 1-4) we present the optimal policy rules and the steady-state pollution stock, as well as the regulator's policies and the firm's control variables, output, q, and abatement, w, as well as the net emissions, e, evaluated at the steady-state pollution stock.

First, we consider the environmental damage coefficient d: 0.01, 0.025, 0.025, 0.35 and 0.05.

Table 1: Sensitivity analysis of the optimal policies and firm's controls with respect to changes in parameter d.

	$d = 0.01$	$d = 0.015$	$d = 0.025$	$d = 0.035$	$d = 0.05$
$\tau^{I^*}(x)$	$83.20 + 0.12x$	$129.15 + 0.15x$	$179.06 + 0.20x$	$207.26 + 0.25x$	$233.38 + 0.30x$
$v^{I^*}(x)$	$377.96 + 0.09x$	$411.72 + 0.11x$	$448.78 + 0.15x$	$469.87 + 0.19x$	$489.50 + 0.23x$
x^{SS}	2298.3	1544.03	932.18	667.62	468.27
$\tau^{I^*}(x^{SS})$	351.46	360.70	368.60	372.19	374.99
$v^{I^*}(x^{SS})$	578.27	585.47	591.73	594.61	596.88
$q^{I^*}(x^{SS})$	212.71	210.50	208.70	207.92	207.33
$w^{I^*}(x^{SS})$	196.31	197.83	199.06	199.59	199.99
$e^{I^*}(x^{SS})$	11.49	7.72	4.66	3.34	2.34

An easy comparison of the emission tax and the abatement subsidy for the different entries in Table 1 allows us to conclude that as the environmental damage parameter increases both the intersection point with the vertical axis and the slope of the tax and the subsidy increase, and hence, the tax and the subsidy increase too for any level of the pollution stock. The steady-state of the pollution stock decreases as parameter d increases, however this fall is more than compensated by the increase in the regulator's optimal tax and subsidy rules, implying that at the steady state both instrument augment with d. Concerning the firm's instruments at the steady-state, any increase of the environmental damage parameter reduces output and augments abatement, and consequently net emissions are reduced. As expected more damages imply higher taxes and subsidies leading to less net and gross emissions and a lower steady-state pollution stock.

Second, we evaluate the effect of changes in the abatement spillover parameter β and consider the following five values, β: 0, 0.015, 0.025, 0.035 and 0.05.

For any value of the abatement spillover parameter presented in Table 2, the optimal

Table 2: Sensitivity analysis of the optimal policies and firm's controls with respect to changes in parameter β.

	$\beta = 0$	$\beta = 0.015$	$\beta = 0.025$	$\beta = 0.035$	$\beta = 0.05$
$\tau^{I^*}(x)$	$190.98 + 0.205x$	$183.81 + 0.204x$	$179.06 + 0.203x$	$174.33 + 0.202x$	$167.28 + 0.201x$
$v^{I^*}(x)$	$445.37 + 0.151x$	$447.45 + 0.152x$	$448.78 + 0.153x$	$450.06 + 0.154x$	$451.90 + 0.156x$
x^{SS}	950.87	939.62	932.18	924.78	913.76
$\tau^{I^*}(x^{SS})$	385.94	375.51	368.60	361.74	351.53
$v^{I^*}(x^{SS})$	588.79	590.61	591.73	592.80	594.28
$q^{I^*}(x^{SS})$	202.85	206.37	208.70	211.01	214.45
$w^{I^*}(x^{SS})$	198.10	198.69	199.06	199.41	199.89
$e^{I^*}(x^{SS})$	4.75	4.70	4.66	4.62	4.57

tax is always positive for any value of the pollution stock. As the spillover parameter augments the optimal tax is reduced and the optimal subsidy is increased regardless of the value of the pollution stock (both the intersection point with the vertical axis and the slope of the tax (subsidy) decrease (increase)). The steady-state of the pollution stock decreases as parameter β increases. This fall in the long-run pollution stock leads to a lower tax and a greater output at the steady state. Concerning the optimal subsidy on abatement and the optimal abatement level the upward shift of these two functions for any value of the pollution stock with an increase in parameter β more than compensates for the fall in the pollution stock at the steady state, leading to an increase of the subsidy and the abatement in the long term. The total effect of an increase in output and in abatement is a fall of net emissions at the steady-state. Notice that in comparison with the case of a variation in d, a variation in β, affects the optimal policy rules in a different way doing compatible a reduction in net emissions with an increase in gross emissions. Nevertheless, it could be pointed out that the variations in relative terms are small.

Third, we focus on the effect of the abatement efficiency parameter (γ). We consider the following five different values of γ: 2.5, 2.75, 3, 3.25 and 3.5.

Table 3 states that as γ increases and firms operate with higher abatement costs,

Table 3: Sensitivity analysis of the optimal policies and firm's controls with respect to changes in parameter in parameter γ.

	$\gamma = 2.5$	$\gamma = 2.75$	$\gamma = 3$	$\gamma = 3.25$	$\gamma = 3.5$
$\tau^{I^*}(x)$	$133.81 + 0.197x$	$157.69 + 0.200x$	$179.06 + 0.203x$	$198.30 + 0.206x$	$215.71 + 0.208x$
$v^{I^*}(x)$	$419.42 + 0.148x$	$434.92 + 0.151x$	$448.78 + 0.153x$	$461.24 + 0.155x$	$472.51 + 0.157x$
x^{SS}	861.84	898.83	932.18	962.38	989.88
$\tau^{I^*}(x^{SS})$	303.36	337.68	368.60	396.61	422.10
$v^{I^*}(x^{SS})$	547.39	570.72	591.73	610.76	628.08
$q^{I^*}(x^{SS})$	230.68	219.12	208.70	199.26	190.66
$w^{I^*}(x^{SS})$	220.85	209.39	199.06	189.70	181.18
$e^{I^*}(x^{SS})$	4.31	4.49	4.66	4.81	4.94

both regulator's optimal policies increase for any value of the pollution stock (both the intersection point with the vertical axis and the slope of the policies increase with γ) as occurs with an increase in d. But now the higher γ, the higher is the steady-state of the pollution stock. Consequently, both the long-run emission tax and subsidy on abatement increase too. The stricter tax policy and laxer subsidy policy as γ becomes higher reduce the output and the abatement level, being the later effect stronger than the former and hence, implying a rise in net emissions.

Fourth, we analyze the effect of the industry competition measured by the number of firms on the optimal regulatory rules and firms' decisions. We start from the base case of a duopoly ($n = 2$) and increase the number of firms in the industry, n: 3, 5, 7 and 9.

Table 4 shows that as industry competition increases both the optimal tax on emissions and the optimal subsidy on abatement decrease, because for both policy rules the ordinate at the origin and the slope decrease as the number of firms in the industry increases. However, there is a case for which this does not occur, when comparing the duopoly with the triopoly. In this case, the intersection point with the vertical increases. However, this increase is not enough as to yield a higher tax when the number of firms goes from 2 to 3 and the steady-state tax decreases when the number of firms augments.

Table 4: Sensitivity analysis of the optimal policies and firm's controls with respect to changes in parameter in parameter n.

	$n=2$	$n=3$	$n=5$	$n=7$	$n=9$
$\tau^{I^*}(x)$	$179.06+0.203x$	$215.18+0.148x$	$166.55+0.107x$	$120.63+0.087x$	$87.90+0.074x$
$v^{I^*}(x)$	$448.78+0.153x$	$436.58+0.121x$	$340.31+0.100x$	$267.60+0.088x$	$216.38+0.081x$
x^{SS}	932.18	755.25	527.59	390.07	299.85
$\tau^{I^*}(x^{SS})$	368.60	326.80	223.08	154.65	110.17
$v^{I^*}(x^{SS})$	591.73	527.93	392.85	302.08	240.55
$q^{I^*}(x^{SS})$	208.70	175.99	134.05	108.03	90.29
$w^{I^*}(x^{SS})$	199.06	165.21	120.91	93.45	74.96
$e^{I^*}(x^{SS})$	4.66	2.52	1.06	0.56	0.33
$Q^{I^*}(x^{SS})$	417.39	527.97	670.26	756.21	812.59
$\Omega^{I^*}(x^{SS})$	398.12	495.63	604.53	654.18	674.66
$E^{I^*}(x^{SS})$	9.32	7.55	5.28	3.90	3.00

The decrease in the tax and in the subsidy comes with a reduction in the steady-state pollution stock. A laxer tax policy and less generous subsidy as competition increases lead to lower levels of output, abatement and net emissions in the long run for firms. Thus, as we already pointed out in Subsection 4.1, competition is good for the environment, it reduces net emissions and the pollution stock. Moreover, we should highlight that competition also increases the total output and the total abatement of the industry, although the individual production and abatement decrease. The total output for the duopoly is 417.4 and when the industry is formed by 9 firms is equal to 812.6. The total abatement is 398.1 for the duopoly, but it is equal to 674.7 when there are 9 firms in the industry. More firms in the industry means more production and more abatement.

4.3 Tax-subsidy scheme II

The efficient outcome can also be implemented combining a tax on production with a subsidy on abatement costs, however the use of a different policy-mix implies that the value function of the firm changes. Now, the HJB equation of the firm is

$$rV^{II}(x) = (a - nq)q - cq - \tau q - (1 - v)\frac{\gamma}{2}w^2 + (V^{II})'(x)(n(q - (1 + \beta(n - 1)w) - \delta x),$$

so that if we substitute the subsidy and the tax by expressions (13) and (25) and the production and the abatement by the efficient strategies given by (14) and (15), we obtain the following differential equation

$$\begin{aligned} rV^{II}(x) &= \frac{1}{n^2}(s + W'(x))^2 + \left(\frac{n-1}{n} + \frac{(2n-1)(1+\beta(n-1))^2}{2\gamma}\right)(V^{II})'(x)W'(x) \\ &+ \frac{(n-1)s - \delta n x}{n}(V^{II})'(x). \end{aligned} \tag{32}$$

For solving this equation, we propose a quadratic specification

$$V^{II}(x) = \frac{A_f^{II}}{2}x^2 + B_f^{II}x + C_f^{II},$$

for which $(V^{II})'(x) = A_f^{II}x + B_f^{II}$.[20] Substituting $V^{II}(x)$, $(V^{II})'(x)$ and $W'(x)$ into (32) gives a system of Riccati equations whose solution for the first two coefficients is

$$A_f^{II} = \frac{2A_r^2}{n^2(r + 2\delta - 2\eta A_r)} > 0, \tag{33}$$

$$B_f^{II} = \frac{2(n(s + B_r)(r + 2\delta) + ((n-1)s - n\eta(2s + B_r))A_r)A_r}{n^3(r + 2\delta - 2\eta A_r)(r + \delta - \eta A_r)} < 0, \tag{34}$$

where

$$\eta = \frac{n-1}{n} + \frac{(2n-1)(1+\beta(n-1))^2}{2\gamma} > 0, \tag{35}$$

and $A_r < 0$ is given by (18).[21] Then, substituting $(V^{II})'(x)$ and $W'(x)$ in (25) using the coefficients of the value functions, we can calculate the optimal tax.

Proposition 4 *The optimal tax is defined by the following rule*

$$\tau^{II*}(x) = -\frac{1}{n}(s + (n+1)B_r - nB_f^I) + \frac{2(n(n+1)\eta + 1)A_r^2 - n(n+1)(r + 2\delta)A_r}{n^2(r + 2\delta - 2\eta A_r)}x, \tag{36}$$

where A_r is negative. The tax increases with the pollution stock, but it can be negative for low values of the pollution stock.

[20] Again the subscript f is used to represent the coefficients of the firm's value function, but now the superscript II stands for the tax-subsidy scheme with a subsidy on abatement costs.

[21] We show that B_f^{II} is negative in the Appendix.

As occurs when the subsidy applies on the abatement effort and for the same reasons the optimal policy could consist of fixing a subsidy on production for low values of the pollution stock. Finally, eliminating $(V^{II})'(x)$ and $W'(x)$ in (13) using the coefficients of the value functions, we obtain the optimal subsidy.

Proposition 5 *The optimal subsidy is given by the following rule*

$$1 - v^{II*}(x) = \frac{2A_r^2 x + B_f^{II} n^2 (r + 2\delta - 2\eta A_r)}{n^2 (r + 2\delta - 2\eta A_r)(A_r x + B_r)}. \tag{37}$$

For $x \leq x_e < x_v = -B_f^{II} n^2 (r + 2\delta - 2\eta A_r)/(2A_r^2)$, $v^{II*}(x)$ *increases with the pollution stock in a nonlinear way in the interval* $[0, 1]$.

Proof. See the Appendix. ∎

Observe, that although the policy game we have proposed is a LQ differential game, in this case the subsidy rule is not linear. For the baseline case we have proposed in the previous subsection, we have calculated the optimal subsidy and we have plotted in Fig. 1. In this figure, we see that the subsidy rule is a strictly concave function of the pollution stock, and that $v^{II*}(x) = (298.06 + 0.129x)/(482.46 + 0.107x) \in (0, 1)$ for all $x \in [0, x_e]$. Thus, we find that the subsidy on abatement costs increases with the pollution stock, but at a decreasing rate.

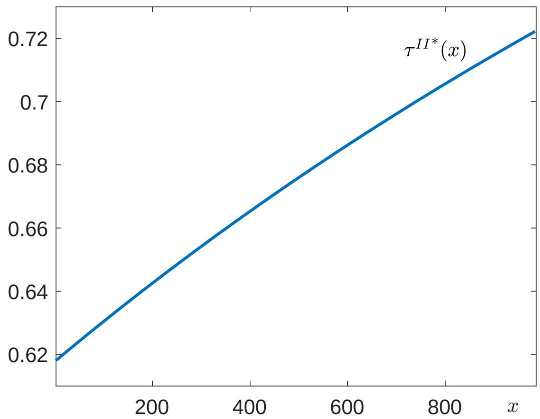

Figure 1: Optimal subsidy rule, scheme II. Baseline case.

We have also carried out the sensitivity analysis for the tax-subsidy scheme II, but we only report here the cases for which we have found qualitative differences with the results obtained for the tax-subsidy scheme I. For the rest of cases we have found the same qualitative results. For the lowest values of d and γ, the intersection point with the vertical axis of the tax rule is negative. Thus, in these cases the optimal policy consists of subsidizing the production for low values of the pollution stock, although the taxes at the steady state are positive. These results suggest that we should expect that the lower the environmental damage parameter, d, and the parameter of the abatement costs, γ, the higher the chances that the tax rule crosses the vertical axis at a negative value. With low values of d the market distortion caused by the market power of firms can be more serious that the one caused by the environmental externality, and then we should expect that the tax becomes a subsidy. The same argument applies for low values of γ. In this case, the abatement technology is very efficient and emissions can be reduced at a low cost making less severe the market distortion caused by the externality. We also find differences in the effects that the different parameters have on the subsidy. Whereas an increase in d and β augments the subsidy when it is applied on the abatement effort, it reduces the subsidy on the abatement costs. Finally, in the case of the number of firms the opposite occurs. More firms in the market reduce the subsidy on the abatement effort, but increase the subsidy on abatement costs. Thus, we have not only that two different incentive structures can implement the same outcome, but that they respond in a different way to changes in the parameter values of the model.

4.4 Comparison of the tax-subsidy schemes

Although, as we have just commented, the two schemes implement the efficient solution we expect that they yield differences in fiscal terms. In this subsection we try to assess these differences. First, next proposition evaluates the effect on the tax rule of using a different subsidy.

Proposition 6 *The optimal taxes for the two tax-subsidy schemes compare as follows,*

$$\tau^{I^*}(x) > \tau^{II^*}(x) \quad \textit{for any } x \geq 0.$$

Proof. See the Appendix. ■

The proposition establishes that for any value of the pollution stock, the optimal tax when the subsidy is on the abatement effort is greater than when it is on abatement costs. In the proof of this result we show that both the slope of the optimal rule and the intersection with the vertical axis when a subsidy is applied on abatement costs is lower than when a subsidy is directly applied on abatement. As both tax-subsidy schemes implement the efficient solution, the steady-state pollution stock will be the same and consequently the steady-state tax will be higher when a subsidy on abatement is applied that when the subsidy is on abatement costs. Therefore, as a direct consequence of the proposition we can conclude that

Corollary 1 *The tax revenues at the steady state are higher when the subsidy is on the abatement effort that when it is on abatement costs.*

However, from a fiscal point of view what is relevant is the fiscal balance, i.e. the difference between the tax revenues and the subsidy expenses. For the example at hand we can compute the fiscal balance for the two schemes. For the baseline case if a subsidy on abatement is applied the fiscal balance is given by the following expression:

$$\tau^{I^*}(x)q^*(x) - v^{I^*}(x)w^*(x) = -0.017x^2 + 1.245x - 27640.5.$$

It can be easily shown that the expression above takes always negative values for any value of the pollution stock, and therefore, there is always a fiscal deficit.

For the baseline case if instead a subsidy on abatement costs is applied the fiscal balance reads:

$$\tau^{II^*}(x)q^*(x) - v^{II^*}(x)\frac{\gamma}{2}(w^*(x))^2 = -0.0092x^2 + 5.016x - 15012.8.$$

In this case there is a fiscal deficit too.

Finally, if we compare the fiscal balances at the steady state we see that the fiscal deficit is lower when the subsidy is on abatement costs which means that subsidy expenses are lower in this case, since we have already showed that the tax revenues are also lower. Thus, when the tax-subsidy scheme II is used the government will collect less taxes and expend less money on subsidies to the firms resulting in a negative fiscal balance that is lower than the fiscal deficit the government will obtain applying the tax-subsidy scheme I. Then, if the criterion for choosing the type of subsidy by the regulator is the one that generates the most favorable fiscal balance, the regulator will choose to subsidize the abatement costs.

5 Conclusions

This paper studies an efficiency-inducing policy for a polluting oligopoly when pollution abatement is technologically feasible, there are spillovers coming from the firms' abatement effort, and environmental damages depend on the pollution stock. Using a dynamic policy game between the regulator and the oligopolists, we show that a tax-subsidy scheme can implement the efficient outcome as a regulated market equilibrium. The scheme consists of a combination of a tax on production and a subsidy. For the subsidy we consider two alternatives. A subsidy on the abatement effort and a subsidy on abatement costs. Both schemes yield a different tax rule, but both implement the efficient outcome. We find that if firms act strategically taking into account when they decide on abatement and production the evolution of the pollution stock, they will internalize the positive externality associated with the spillovers. Firms are interested in the dynamics of the pollution stock because they expect that the levels of the policy instruments selected by the regulator will depend on the level of this variable. But as the dynamics of the pollution stock depends on the total emissions, firms will take into account the effects of their decision on abatement on other firms' abatement as long as they affect total emissions and the spillovers will be internalized by firms. In this case, we have that the subsidy only reflects the divergence between the social and private valuation of the pollution stock associated with the decision on abatement, and consequently, the tax has to correct the two market failures associated with production, the market power of firms and the negative externality caused by pollution. Thus, the tax could be negative if the first distortion dominates the second. Nevertheless, if the main distortion in the market allocation is the one caused by pollution, the efficiency-inducing policy will consist of a tax on production and a subsidy either on abatement effort or on abatement costs. Although both policies implement the efficient outcome they yield different fiscal balances. Using a LQ policy game we find that the application of a subsidy on abatement costs relaxes the tax rule, interestingly it also yields a lower fiscal deficit at the steady state. A numerical exercise show that both tax-subsidy schemes present a negative balance at the steady state for all parameter values we have studied, but when a subsidy on abatement costs is applied the fiscal deficit is always lower. Thus, our policy recommendation is that from a fiscal perspective a subsidy on abatement costs should be adopted instead of a subsidy on abatement.

A limitation of our approach is that it is assumed an emission function that is additively separable in production (gross emissions) and abatement. To overcome this limitation, a possibility would be to consider green innovation that could reduce the emissions to output ratio or the abatement costs and evaluate also in this framework the effects of spillovers on innovation.[22] We could also consider that the R&D capital can be adjusted through investment.[23] In this case, we could analyze the dynamic interdependence between the accumulation of emissions and the investment in R&D. A further step in this line of research would be to characterize the optimal environmental policy when the pollution stock or the R&D capital are subject to a stochastic evolution.[24] Finally, another interesting issue to address would be to investigate which would be the optimal environmental policy with free entry in the market. All these questions are part of our research agenda.

[22]Two recent papers addressing this issue are Langinier and Chaudhuri (2020) and Masoudi (2022). In both papers the R&D investment reduces the coefficient emissions/production. Langinier and Chaudhuri (2020) investigate the impact of patent policies and emission taxes on green innovation, and on the emission level in the presence of green consumers. Masoudi (2022) characterizes an efficiency-inducing policy consisting of a tax on emissions and a subsidy on R&D investment in a static setting and focusing on competitive firms.

[23]See the paper by Martín-Herrán and Rubio (2018b) for the case of a polluting monopoly that invests in an abatement technology.

[24]Borrero (2022) addresses this issue for the case of a polluting oligopoly when firms can use an abatement technology and there exists uncertainty in the evolution of the pollution stock.

Appendix

Proof of Proposition 3

As the subsidy increases with the pollution stock, we can conclude that the subsidy is is guaranteed to be positive for all $x \geq 0$ if the intersection with the vertical axis, $(1+\beta(n-1))(B_f^I - B_r)$, is positive too, a condition that will be satisfied if $B_f^I - B_r > 0$. According to (28) this difference is given by the following expression

$$B_f^I - B_r = \frac{n\gamma(r+2\delta)(2s\gamma + hB_r)A_r + (n-1)(s\gamma(h-4g) - ghB_r)A_r^2}{n(n\gamma(r+\delta)-(n-1)gA_r)(n\gamma(r+2\delta)-2(n-1)gA_r)} - B_r.$$

Next, we redefine g and h as follows

$$g = n\varepsilon^2 + \gamma, \quad h = n^2\varepsilon^2 + 2\gamma, \tag{38}$$

where

$$\varepsilon = (1+\beta(n-1))^2, \tag{39}$$

and rewrite the difference accordingly

$$\begin{aligned} & B_f^I - B_r \qquad (40) \\ = \; & \frac{n\gamma(r+2\delta)(2s\gamma + (n^2\varepsilon^2+2\gamma)B_r)A_r}{n(n\gamma(r+\delta)-(n-1)(n\varepsilon^2+\gamma)A_r)(n\gamma(r+2\delta)-2(n-1)(n\varepsilon^2+\gamma)A_r)} \\ + \; & \frac{(n-1)(s\gamma(n(n-4)\varepsilon^2 - 2\gamma) - (n^2\varepsilon^2+2\gamma)(n\varepsilon^2+\gamma)B_r)A_r^2}{n(n\gamma(r+\delta)-(n-1)(n\varepsilon^2+\gamma)A_r)(n\gamma(r+2\delta)-2(n-1)(n\varepsilon^2+\gamma)A_r)} \\ - \; & \frac{B_r n(n\gamma(r+\delta)-(n-1)(n\varepsilon^2+\gamma)A_r)(n\gamma(r+2\delta)-2(n-1)(n\varepsilon^2+\gamma)A_r)}{n(n\gamma(r+\delta)-(n-1)(n\varepsilon^2+\gamma)A_r)(n\gamma(r+2\delta)-2(n-1)(n\varepsilon^2+\gamma)A_r)}. \end{aligned}$$

As the denominator in the expressions above is positive because A_r is negative, we focus on the sign of the numerator. Substituting B_r by (19) and developing the numerator yields

$$\frac{s\gamma A_r(f_0A_r^2 + f_1A_r + f_2)}{(n\varepsilon^2+\gamma)A_r - (r+\delta)\gamma} > 0,$$

since

$$\begin{aligned} f_0 &= (n-1)n(2\left(n^2-2\right)n\varepsilon^4 + 2\left(2n^2-n-2\right)\gamma\varepsilon^2 + 2\left(n-1\right)\gamma^2) > 0, \\ f_1 &= -\gamma(n^3(n-2)(r+2\delta)\varepsilon^2 + n\left(n-1\right)\left(2n^2+n-4\right)(r+\delta)\varepsilon^2 \\ &+ n^2(n-1)(r+2\delta)\gamma + 2\left(n+1\right)\left(n-1\right)^2(r+\delta)\gamma) < 0, \\ f_2 &= n(r+2\delta)(r+\delta)\gamma^2\left(n^2-2\right) > 0, \end{aligned}$$

for $n \geq 2$. Thus, the numerator of (41) is positive and hence the difference $B_f^I - B_r$ is positive too and we can conclude that the subsidy on abatement effort is positive for all $x \geq 0$.

Sign of coefficient B_f^{II}

The sign of coefficient B_f^{II} according to (34) depends on the sign of the following expression (one of the factors in the numerator) since the denominator is positive because A_r is negative

$$n(s + B_r)(r + 2\delta) + ((n-1)s - n\eta(2s + B_r))\, A_r. \tag{41}$$

Substituting B_r by (19) using the auxiliary expressions g as ε given in (38) and (39) expression (41) reads:

$$\begin{aligned} &\frac{s}{\gamma(\delta + r) - (\gamma + n\varepsilon^2)A_r} \left(n\gamma(\delta + r)(r + 2\delta) - n^2\varepsilon^2(r + 2\delta)A_r\right. \\ &\left.+\gamma(n - 1 - 2n\eta)(\delta + r)A_r + (n\eta(\gamma + 2n\varepsilon^2) - (n-1)(\gamma + n\varepsilon^2))A_r^2\right). \end{aligned} \tag{42}$$

Now, we use (35) to eliminate η resulting in

$$n - 1 - 2n\eta = -\left((n-1) + \frac{n(2n-1)}{\gamma}\varepsilon^2\right) < 0,$$

and

$$n\eta(\gamma + 2n\varepsilon^2) - (n-1)(\gamma + n\varepsilon^2) = \left((n-1)n + \frac{n(2n-1)}{2\gamma}(\gamma + 2n\varepsilon^2)\right)\varepsilon^2 > 0.$$

Then, expression (43) is positive because A_r is negative. Consequently (41) is positive, hence, multiplied by A_r gives a negative value for the numerator of (34) and then we can conclude that B_f^{II} is negative.

Proof of Proposition 5

For the subsidy rule defined by (37) the denominator is negative since A_r and B_r are negative. On the other hand, the numerator is an increasing linear function that takes a negative value for $x = 0$, because B_f^{II} is negative. Then, we can conclude that $1 - v^{II^*}(x)$ is positive for all $x < x_v = -B_f^{II}n^2(r + 2\delta - 2\eta A_r)/(2A_r^2)$, where x_v is the pollution stock for which the numerator is null. We have to ensure that $v^{II^*}(x)$ is in the interval $(0,1)$. To show this point, first we check that $v^{II^*}(0)$ belongs to this interval. As $1-v^{II^*}(0)=B_f^{II}/B_r>0$, $v^{II^*}(0)$ must be lower than 1 and it will be higher than 0 if $B_r<B_f^{II}$. The difference between these two coefficients according to (34) is given by

$$\begin{aligned} B_r - B_f^{II} &= B_r - \frac{2(n(s+B_r)(r+2\delta)+((n-1)s-n\eta(2s+B_r))\,A_r)A_r}{n^3(r+2\delta-2\eta A_r)(r+\delta-\eta A_r)} \qquad (43)\\ &= \frac{n^3(r+2\delta-2\eta A_r)(r+\delta-\eta A_r)B_r}{n^3(r+2\delta-2\eta A_r)(r+\delta-\eta A_r)}\\ &- \frac{2(n(s+B_r)(r+2\delta)+((n-1)s-n\eta(2s+B_r))\,A_r)A_r}{n^3(r+2\delta-2\eta A_r)(r+\delta-\eta A_r)}, \end{aligned}$$

where the denominator is positive because A_r is negative. Substituting B_r by (19) and using g and ε given in (38) and (39) in the numerator and developing it gives

$$-\frac{sA_r(k_0A_r^2+k_1A_r+k_2)}{(n\varepsilon^2+\gamma)A_r-(r+\delta)\gamma}<0,$$

since

$$\begin{aligned} k_0 &= (n(n-1)-1)\gamma+\frac{n}{2}(n(2n-1)-4)\varepsilon^2>0,\\ k_1 &= -\left(\frac{n(2n-1)-4}{2}\varepsilon^2+\gamma(n-1)\right)<0,\\ k_2 &= n(n^2-2)(r+2\delta))(r+\delta)\gamma>0, \end{aligned}$$

for $n\geq 2$. Thus, we can establish that (44) is negative that implies $B_r<B_f^{II}$. Then, we can conclude that $v^{II^*}(0)\in(0,1)$. Finally, we calculate the derivative of $1-v^{II^*}(x)$ with respect to the pollution stock

$$\begin{aligned} (1-v^{II^*})'(x) &= \frac{2A_r^2n^2(r+2\delta-2\eta A_r)(A_rx+B_r)}{n^4(r+2\delta-2\eta A_r)^2(A_rx+B_r)^2}\\ &- \frac{(2A_r^2x+B_fn^2(r+2\delta-2\eta A_r))n^2(r+2\delta-2\eta A_r)A_r}{n^4(r+2\delta-2\eta A_r)^2(A_rx+B_r)^2}, \end{aligned}$$

that takes a negative value for $x<x_v$. Then, we have that $v^{II^*}(x)$ must be increasing, but as $1-v^{II^*}(x)$ is positive for all $x<x_v$, $v^{II^*}(x)$ cannot reach a value higher than 1 for $x\in[0,x_v)$. The nonlinearity of the subsidy rule is established directly from (37) since the linear functions of x that appear in the numerator and the denominator have different slopes and intersection with the vertical axis.

Finally, to conclude the proof of the proposition we prove that $x_v>x_e$. First, we rewrite the expression of $x_v=-B_f^{II}n^2(r+2\delta-2\eta A_r)/(2A_r^2)$, substituting the expression of B_f^{II} in (34). After some easy computations x_e and using the expressions of η and g in (35) and (38) reads:

$$x_v = -s\frac{n(\gamma(r+\delta) - n\varepsilon^2 A_r)(r+2\delta-2\eta A_r) + ((n-1)(\gamma(r+\delta)-gA_r)+n\gamma\eta A_r)}{n(\gamma(r+\delta)-A_r)(r+\delta-\eta A_r)A_r}.$$

Takin into account the expression of the threshold x_e in (23) the difference $x_v - x_e$ reads:

$$\begin{aligned}
& s\frac{n\gamma^2(r+\delta)(r+\delta-\eta A_r)}{ng(\gamma(r+\delta)-gA_r)(r+\delta-\eta A_r)A_r} \\
- \quad & s\frac{g\left[n(\gamma(r+\delta)-n\varepsilon A_r)(r+2\delta-2\eta A_r) + A_r\left((n-1)(\gamma(r+\delta)-gA_r)+n\gamma\eta A_r\right)\right]}{ng(\gamma(r+\delta)-gA_r)(r+\delta-\eta A_r)A_r}
\end{aligned}$$

The denominator is negative because $A_r < 0$, hence the difference $x_v - x_e$ is positive if and only the numerator is negative too. The numerator can be rewritten as a second-order polynomial in variable A_r as follows:

$$l_0 A_r^2 + l_1 A_r + l_2,$$

with

$$\begin{aligned}
l_0 &= -g\left(\gamma + \gamma(\eta-1)n + n\epsilon^2((2\eta-1)n+1)\right), \\
l_1 &= (\delta+r)\left(n^3\epsilon^4 + \gamma^2((\eta-1)n+1) + \gamma n\epsilon^2(2\eta n+1)\right) + \delta n^2\epsilon^2 g, \\
l_2 &= -\gamma n(\delta+r)\left(\gamma\delta + n\epsilon^2(2\delta+r)\right) < 0.
\end{aligned}$$

We can conclude that $l_0 A_r^2 + l_1 A_r + l_2 < 0$, and consequently, $x_v - x_e > 0$, because as shown below $l_0 < 0$ and $l_1 > 0$ for any $n \geq 2$, once the expressions of η, g and ϵ given in (35), (38) and (39) are substituted:

$$\begin{aligned}
l_0 &= -g(2n(2n-1)(\beta(n-1)+1)^2 + \gamma(4n-3)), \\
l_1 &= n(\beta(n-1)+1)^2\left(\delta n\left(\gamma + n(\beta(n-1)+1)^2\right) + \frac{1}{2}(\delta+r)\left(2n(3n-1)(\beta(n-1)+1)^2\right.\right. \\
&+ \left.\left. 3\gamma(2n-1)\right)\right).
\end{aligned}$$

Proof of Proposition 6

We begin with the comparison of the slope of the tax rules. From (29) and (36) we know that the difference in the slopes is given by

$$\frac{(h+2(n+1)(n-1)g)A_r^2 - (n+1)\gamma(r+2\delta)A_r}{n^2\gamma(r+2\delta) - 2(n-1)ngA_r}$$

$$-\frac{2(n(n+1)\eta+1)A_r^2-n(n+1)(r+2\delta)A_r}{n^2(r+2\delta-2\eta A_r)}$$

that yields

$$= \frac{n^2((h+2(n+1)(n-1)g)A_r^2-(n+1)\gamma(r+2\delta)A_r)(r+2\delta-2\eta A_r)}{n^3(n\gamma(r+2\delta)-2(n-1)gA_r)(r+2\delta-2\eta A_r)}$$
$$-\frac{(n^2\gamma(r+2\delta)-2(n-1)ngA_r)(2(n(n+1)\eta+1)A_r^2-n(n+1)(r+2\delta)A_r)}{n^3(n\gamma(r+2\delta)-2(n-1)gA_r)(r+2\delta-2\eta A_r)}, \quad (44)$$

where the denominator is positive because A_r is negative. Developing the numerator and simplifying terms we obtain the following expression

$$nA_r^2(n(h-2\gamma)(r+2\delta)+2(2(n-1)g-nh\eta)A_r), \quad (45)$$

where

$$h-2\gamma = n^2(1+\beta(n-1))^2 > 0,$$
$$2(n-1)g-nh\eta = -\frac{n}{2\gamma}(1+\beta(n-1))^2(n^2(2n-1)(1+\beta(n-1))^2+2(n^2-n+1)\gamma) < 0.$$

Therefore, (45) is positive and we can conclude that the slope of the optimal tax rule when a subsidy is applied on abatement costs is lower than the slope of the optimal tax rule when a subsidy is applied directly on abatement.

On the other hand, the difference in the intersection point with the vertical axis is given by the difference $B_f^I - B_f^{II}$. The intersection point with the vertical axis for scheme I is greater than for scheme II if and only if $B_f^I > B_f^{II}$. From (28) and (34), one has

$$B_f^I - B_f^{II} = \frac{A_r(\Delta_1-\Delta_2)}{n^3},$$

where

$$\Delta_1 = \frac{n^2(A_r(n-1)(\gamma s(h-4g)-B_r gh)+\gamma n(2\delta+r)(B_r h+2\gamma s))}{(\gamma n(\delta+r)-A_r g(n-1))(\gamma n(2\delta+r)-2A_r g(n-1))},$$
$$\Delta_2 = \frac{2(A_r B_r \eta n+n(B_r+s)(-2A_r\eta+2\delta+r)+A_r(n-1)s)}{(-2A_r\eta+2\delta+r)(-A_r\eta+\delta+r)}.$$

Because $A_r<0$, the sign of the difference $B_f^I - B_f^{II}$ is the opposite to the sign of the difference $\Delta_1-\Delta_2$.

The difference $\Delta_1-\Delta_2$ can be rewritten as:

$$\Delta_1-\Delta_2 = \frac{Num(\Delta_1-\Delta_2)}{(-2A_r\eta+2\delta+r)(-A_r\eta+\delta+r)(\gamma n(\delta+r)-A_r g(n-1))(\gamma n(2\delta+r)-2A_r g(n-1))},$$

where

$$
\begin{aligned}
Num(\Delta_1 - \Delta_2) &= (-2A_r\eta + 2\delta + r)(-A_r\eta + \delta + r)Num(\Delta_1) \\
&- (\gamma n(\delta + r) - A_r g(n-1))(\gamma n(2\delta + r) - 2A_r g(n-1))Num(\Delta_2),
\end{aligned}
$$

with

$$
\begin{aligned}
Num(\Delta_1) &= n^2(A_r(n-1)(\gamma s(h-4g) - B_r gh) + \gamma n(2\delta + r)(B_r h + 2\gamma s)), \\
Num(\Delta_2) &= 2(A_r B_r \eta n + n(B_r + s)(-2A_r\eta + 2\delta + r) + A_r(n-1)s).
\end{aligned}
$$

The denominator of $\Delta_1 - \Delta_2$ is positive because A_r is negative, and hence, the sign of the difference $\Delta_1 - \Delta_2$ is the same as the sign of its numerator, $Num(\Delta_1 - \Delta_2)$. Substituting the expression of B_r given in (19), and after some simplifications $Num(\Delta_1 - \Delta_2)$ can be rewritten as

$$
Num(\Delta_1 - \Delta_2) = \frac{A_r(\Lambda_1 + \Lambda_2 A_r + \Lambda_3 A_r^2 + \Lambda_4 A_r^3)}{A_r g - \gamma(\delta + r)}, \tag{46}
$$

where

$$
\begin{aligned}
\Lambda_1 &= -\gamma^2 n^2 s(\delta + r)(2\delta + r)[(2\delta + r)(hn + 2g(n-1)) + (\delta + r)(2\gamma + h(n-1)) \\
&- 2\gamma n(\delta(2\eta + 3) + (\eta + 2)r)], \\
\Lambda_2 &= \gamma n s[2g(n-1)\left(hn(\delta + r)(2\delta + r) + \gamma\left(2\delta^2(2-7n) + (3-7n)r^2 + \delta(7-20n)r\right)\right) \\
&- \gamma\eta n(h(4\delta + 3r)(\delta(1-3n) + (1-2n)r) + 2\gamma(2\eta + 1)n(\delta + r)(2\delta + r)) \\
&+ 2g^2(n-1)(2\delta + r)((3n-2)r - 2\delta(1-2n)) - 2\gamma\eta g n^2(2\delta + r)^2], \\
\Lambda_3 &= 2s\left[\gamma\eta g n^2((n-1)(\gamma - h)(4\delta + 3r) + 2\gamma\eta(-2\delta + 4\delta n + 3nr - 2r))\right. \\
&- \gamma^2 h\eta^2 n^2((n-1)(\delta + r) + n(2\delta + r)) - 2g^3(n-1)^2 n(2\delta + r) \\
&- \left.\gamma g^2(n-1)^2(2(\delta + \delta(2\eta - 5)n) + r((4\eta - 7)n + 2))\right], \\
\Lambda_4 &= 4g(n-1)s\left[\gamma h\eta^2 n^2 + g^2(n-1)((2\eta - 1)n + 1) - \gamma\eta g n(n - 1 + 2\eta n)\right].
\end{aligned}
$$

Taking into account that A_r is negative, the sign of $Num(\Delta_1 - \Delta_2)$ in expression (46) coincides with the sign of the third-order polynomial in variable A_r, $\Lambda_1 + \Lambda_2 A_r + \Lambda_3 A_r^2 + \Lambda_4 A_r^3$. From the expression of Λ_1 it is clear that Λ_1 is negative for any $n \geq 2$, because both h and η are positive. However, to completely characterize the sign of coefficients

$\Lambda_2, \Lambda_3, \Lambda_4$, we need to substitute the expressions of g, h and η given in (38) and (35). After the substitution coefficient Λ_4 reads

$$\begin{aligned}\Lambda_4 &= \frac{1}{\gamma}(n-1)ns(\beta(n-1)+1)^2\left(\gamma+n(\beta(n-1)+1)^2\right)\left[\left(2\gamma n\left((n-1)\left(4n^2-2n-1\right)\right)\right.\right.\\ &\quad \left.\left.(\beta(n-1)+1)^2+n^2(n(2n-3)(2n+1)+2)(\beta(n-1)+1)^4+2\gamma^2(n-1)(2(n-1)n+1)\right)\right].\end{aligned}$$

Therefore, Λ_4 is positive for any $n \geq 2$.

Unfortunately, the expressions for coefficients Λ_2 and Λ_3 are much longer and more complicated:

$$\begin{aligned}\Lambda_2 &= \gamma ns\Big[2(n-1)(2\delta+r)\left(\gamma+n(\beta(n-1)+1)^2\right)^2((3n-2)r-2\delta(n-2))\\ &+ 2(n-1)\left(\gamma+n(\beta(n-1)+1)^2\right)\left(n(\delta+r)(2\delta+r)\left(2\gamma+(\beta(n-1)n+n)^2\right)\right.\\ &+ \gamma(-n)(\delta+r)(2\delta+r)-\gamma(4\delta+3r)(\delta(3n-1)+(2n-1)r))\\ &+ \frac{1}{2}\left(n(2n-1)(\beta(n-1)+1)^2+2\gamma(n-1)\right)\left(-2n(2\delta+r)^2\left(\gamma+n(\beta(n-1)+1)^2\right)\right.\\ &- 2(\delta+r)(2\delta+r)\left(n(2n-1)(\beta(n-1)+1)^2+\gamma(3n-2)\right)\\ &+ \left.(4\delta+3r)\left(2\gamma+(\beta(n-1)n+n)^2\right)(\delta(3n-1)+(2n-1)r)\right)\Big],\\ \Lambda_3 &= -\frac{1}{2}s\left(2\gamma+(\beta(n-1)n+n)^2\right)\left(n(2n-1)(\beta(n-1)+1)^2+2\gamma(n-1)\right)^2[(n-1)(\delta+r)\\ &+ n(2\delta+r)]-4(n-1)^2ns(2\delta+r)\left(\gamma+n(\beta(n-1)+1)^2\right)^3+s\left(\gamma+n(\beta(n-1)+1)^2\right)\\ &\quad \left(n(2n-1)(\beta(n-1)+1)^2+2\gamma(n-1)\right)\left((1-n)n(4\delta+3r)\left(\gamma+(\beta(n-1)n+n)^2\right)\right.\\ &- \left.\left((1-2n)n(\beta(n-1)+1)^2-2\gamma(n-1)\right)(2\delta(2n-1)+(3n-2)r)\right)\\ &- 2(1-n)(n-1)s\left(\gamma+n(\beta(n-1)+1)^2\right)^2\left(2(\delta+r)\left((1-2n)n(\beta(n-1)+1)^2-2\gamma(n-1)\right)\right.\\ &+ \left.\gamma(2\delta(5n-1)+(7n-2)r)\right).\end{aligned}$$

Given the complexity of the above expressions for Λ_2 and Λ_3, we have resorted to studying their sign with the help of the Reduce command of the mathematical software Mathematica. This command reduces expressions by solving both equations and inequalities by eliminating quantifiers. This command allows us to determine that Λ_2 and Λ_3 are positive and negative, respectively, for any $n \geq 2$. Therefore, we can conclude that the third-order polynomial in variable A_r, $\Lambda_1 + \Lambda_2 A_r + \Lambda_3 A_r^2 + \Lambda_4 A_r^3$ is always negative for any $A_r < 0$, and consequently, $B_f^I - B_f^{II} > 0$.

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